3F: Duality

Defn 3.92 A <u>linear functional</u> on V is a linear map $\varphi: V \to \mathbb{F}$. In other words, $\varphi \in \mathcal{L}(V, \mathbb{F})$

Eg 3.93 • Define $\varphi: \mathbb{R}^3 \to \mathbb{R}$ by $\quad \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$$\varphi(x, y, z) = 4x - 5y + 2z$$

then φ is a linear functional on $\mathbb{R}^3$

• For some $c_1, \dots, c_n \in \mathbb{F}$, then the map $\varphi: \mathbb{F}^n \to \mathbb{F}$ defined by

$$\varphi(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

then φ is a linear functional on $\mathbb{F}^n$

• Define $\varphi: P(\mathbb{R}) \to$ by

$$\varphi(p) = \int_0^1 p(x)\, dx$$

then φ is a linear functional on $P(\mathbb{R})$

Defn 3.94 The dual space of V, denoted V', is the vector space of all linear functionals on V.
In other words,

$$V' = \mathcal{L}(V, \mathbb{F})$$

Thm 3.95 <u>$\dim V' = \dim V$</u>

Suppose V is finite-dimensional. Then V' is also finite-dimensional and

$$\dim V' = \dim V$$

proof:

$$\begin{aligned} \dim V' &= \dim \mathcal{L}(V, \mathbb{F}) \\ &= (\dim V)(\dim \mathbb{F}) \text{ by 3.61 of Axler} \\ &= (\dim V) \cdot 1 \\ &= \dim V \end{aligned}$$

$\boxtimes$

Defn 3.96 If $v_1, \dots, v_n$ is a basis of V, then the <u>dual basis</u> of $v_1, \dots, v_n$ is the list $\varphi_1, \dots, \varphi_n$ of elements of V' where each φ_j for any $j = 1, \dots, n$ is a linear functional on V that satisfies

$$\varphi_j(v_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

Group Exam Q! (arbitrary basis) $\mathbb{R}^3$

Eg 3.97 What is the dual basis of the standard basis $e_1, \dots, e_n$ of $\mathbb{F}^n$?

$$\begin{aligned} e_1 &= (1, 0, \dots, 0) \\ e_2 &= (0, 1, 0, \dots, 0) \\ e_3 &= (0, 0, 1, 0, \dots, 0) \\ &\vdots \\ e_n &= (0, \dots, 0, 1) \end{aligned}$$

Solution: For all $j = 1, \dots, n$ write

$$\varphi_j(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

$c_1 = 1$ then $1 = \varphi_1(v_1) = \varphi_1(e_1) = \varphi_1(1, 0, \dots, 0) = c_1(1) + c_2(0) + \dots + c_n(0) = c_1$

$c_2 = 0$ $\quad 0 = \varphi_1(v_2) = \varphi_1(e_2) = \varphi_1(0, 1, 0, \dots, 0) = c_1(0) + c_2(1) + c_3(0) + \dots + c_n(0) = c_2$

$\vdots$

$c_n = 0$ $\quad 0 = \varphi_1(v_n) = \varphi_1(e_n) = \varphi_1(0, \dots, 0, 1) = c_1(0) + \dots + c_{n-1}(0) + c_n(1) = c_n$

so $\varphi_1(x_1, \dots, x_n) = \overset{c_1}{(1)} x_1 + \overset{c_2}{(0)} x_2 + \dots + \overset{c_n}{(0)} x_n = x_1$

$$\boxed{\begin{aligned} \text{Similarly} \quad \varphi_1(x_1, \dots, x_n) &= x_1 \\ \varphi_2(x_1, \dots, x_n) &= x_2 \\ \varphi_3(x_1, \dots, x_n) &= x_3 \\ &\vdots \\ \varphi_n(x_1, \dots, x_n) &= x_n \end{aligned}}$$

So $\varphi_1, \dots, \varphi_n$ as defined above is the dual basis of $v_1, \dots, v_n$.

<u>Alter</u>

Define φ_j to be the linear functional on $\mathbb{F}^n$ that selects the j^{th} coordinate of a vector in $\mathbb{F}^n$.
In other words,

$$\varphi_j(x_1, \dots, x_n) = x_j$$

for all $(x_1, \dots, x_n) \in \mathbb{F}^n$.

Thm 3.98 <u>Dual Basis is a basis of the dual space</u>

Suppose V is finite-dimensional, then the dual basis of a basis of V is a basis of V'

proof: Let $v_1, \dots, v_n$ be a basis of V, and let $\varphi_1, \dots, \varphi_n$ be a dual basis of $v_1, \dots, v_n$. We will show that $\varphi_1, \dots, \varphi_n$ is linearly independent. Suppose $a_1, \dots, a_n \in \mathbb{F}$ satisfy

$$a_1\varphi_1 + \dots + a_n\varphi_n = 0$$

Then for any $j = 1, \dots, n$, we have

$$\begin{aligned}(a_1\varphi_1 + \dots + a_n\varphi_n)(v_j) &= a_1\varphi_1(v_j) + \dots + a_n\varphi_n(v_j) \\ &= a_1\varphi_1(v_j) + \dots + a_j\varphi_j(v_j) + \dots + a_n\varphi_n(v_j) \\ &= a_1 \cdot 0 + \dots + a_j \cdot 1 + \dots + a_n \cdot 0 \\ &= a_j\end{aligned}$$

So we have

$$\begin{aligned}a_j &= (a_1\varphi_1 + \dots + a_n\varphi_n)(v_j) \\ &= 0(v_j) \\ &= 0\end{aligned}$$

In other words,

$$a_1 = 0, \dots, a_n = 0$$

So $\varphi_1, \dots, \varphi_n$ is linearly independent.

By 3.95 of Axler, $\dim V' = \dim V$.

By 2.39 of Axler, $\varphi_1, \dots, \varphi_n$ is a basis of V'. ☑

Defn 3.99 If $T \in \mathcal{L}(V, W)$, then the <u>dual map</u> of T is the linear map $T' \in \mathcal{L}(W', V')$ defined by

$$T'(\varphi) = \varphi \circ T$$

input a linear functional

output a linear functional

Show: $T' \in \mathcal{L}(W', V')$

Let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in W'$ be arbitray.

- additivity: $$\begin{aligned}T'(\varphi + \psi) &= (\varphi + \psi) \circ T \\ &= \varphi \circ T + \psi \circ T \\ &= T'(\varphi) + T'(\psi).\end{aligned}$$
- homogeneity: $$\begin{aligned}T'(\lambda\varphi) &= (\lambda\varphi) \circ T \\ &= \lambda(\varphi \circ T) \\ &= \lambda T'(\varphi).\end{aligned}$$

E 3.100 Define $D: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $Dp = p'$.

- Define $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ by $$\varphi(p) = p(3).$$ Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ that satisfies $$\begin{aligned}(D'(\varphi))(p) &= (\varphi \circ D)(p) \\ &= \varphi(Dp) \\ &= \varphi(p') \\ &= p'(3).\end{aligned}$$
- Define $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ by $$\varphi(p) = \int_0^1 p(x)\,dx.$$ Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ given by $$\begin{aligned}(D'(\varphi))(p) &= (\varphi \circ D)(p) \\ &= \varphi(Dp) \\ &= \varphi(p') \\ &= \int_0^1 p'(x)\,dx \\ &= p(1) - p(0).\end{aligned}$$

Thm 3.101 <u>Algebraic Properties of Dual Maps</u>

(a) $(S+T)' = S' + T'$ for all $S, T \in \mathcal{L}(V, W)$

(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$ and for all $T \in \mathcal{L}(V, W)$

(c) $(ST)' = T'S'$ for all $T \in \mathcal{L}(V, V)$ and for all $S \in \mathcal{L}(V, W)$

Proof →

proof: (a) For all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$\begin{aligned}(S+T)'(\varphi) &= \varphi \circ (S+T) \\ &= \varphi \circ S + \varphi \circ T \\ &= S'(\varphi) + T'(\varphi).\end{aligned}$$

(b) For all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$\begin{aligned}(\lambda T)'(\varphi) &= \varphi \circ (\lambda T) \\ &= \lambda \varphi \circ T \\ &= \lambda T'(\varphi).\end{aligned}$$

(c) For all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$\begin{aligned}(ST)'(\varphi) &= \varphi \circ (ST) \\ &= (\varphi \circ S) \circ T \\ &= T'(\varphi \circ S) \\ &= T'(S'(\varphi)) \\ &= (T'S')(\varphi).\end{aligned}$$

So $(ST)'(\varphi) = T'S'$.

Defn 3.102 If U is a subspace of V, then the <u>annihilator</u> of U, denoted U°, is defined by

$$U^\circ = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

Eg 3.103 Let U be the subspace of $\mathcal{P}(\mathbb{R})$ consisting of all polynomial multiples of x^2 such as $x^2 p(x)$.

If $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ is defined by

$$\varphi(p) = p'(0),$$

$$\varphi \in U^\circ$$

$x^2 p(x) \in U$, if $p \in \mathcal{P}(\mathbb{R})$ and

$$\begin{aligned}\varphi(x^2 p(x)) &= (x^2 p(x))'\big|_{x=0} \\ &= \left(2x\,p(x) + x^2 p'(x)\right)\big|_{x=0} \\ &= 2(0)p(0) + 0^2 p'(0) \\ &= 0.\end{aligned}$$

Eg 3.104 Let e_1, e_2, e_3, e_4, e_5 denote the standard basis of $\mathbb{R}^5$, and let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ denote the dual basis of $(\mathbb{R}^5)'$.

Suppose
$$\begin{aligned}U &= \operatorname{span}(e_1, e_2) \\ &= \{(x_1, x_2, 0, 0, 0) \in \mathbb{R}^5 : x_1, x_2 \in \mathbb{R}\}.\end{aligned}$$

Show $U^\circ = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

<u>Solution:</u> Recall from (Eg 3.97) that

$$\varphi_j(x_1, x_2, x_3, x_4, x_5) = x_j \quad \text{for any } j = 1, 2, 3, 4, 5.$$

Suppose we have $\varphi \in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

Then $\varphi = c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$ for some $c_3, c_4, c_5 \in \mathbb{R}$.

For all $(x_1, x_2, 0, 0, 0) \in U$, we have

$$\begin{aligned}\varphi(x_1, x_2, 0, 0, 0) &= (c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(x_1, x_2, 0, 0, 0) \\ &= c_3\varphi_3(x_1, x_2, 0, 0, 0) + c_4\varphi_4(x_1, x_2, 0, 0, 0) \\ &\quad + c_5\varphi_5(x_1, x_2, 0, 0, 0) \\ &= c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\ &= 0.\end{aligned}$$

So $\varphi \in U^\circ$, and so $\operatorname{span}(\varphi_3, \varphi_4, \varphi_5) \subset U^\circ$.

Suppose $\varphi \in U^\circ$. Since $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ is the dual basis of $(\mathbb{R}^5)'$, we can write uniquely as

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$$

for some $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$.

Since $e_1 \in U$, we have

$$\begin{aligned}0 &= \varphi(e_1) \\ &= (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_1) \\ &= c_1\varphi_1(e_1) + c_2\varphi_2(e_1) + c_3\varphi_3(e_1) + c_4\varphi_4(e_1) + c_5\varphi_5(e_1) \\ &= c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\ &= c_1\end{aligned}$$

Since $e_2 \in U$

$$\begin{aligned}0 &= \varphi(e_2) \\ &= (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_2)\end{aligned}$$

$= c_1\varphi_1(e_2) + c_2\varphi_2(e_2) + c_3\varphi_3(e_2) + c_4\varphi_4(e_2) + c_5\varphi_5(e_2)$

$= c_1 \cdot 0 + c_2 \cdot 1 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0$

$= c_2$

Therefore,

$\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$

$= 0\varphi_1 + 0\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$

$= c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$

$\in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

So $U^0 \subset \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

Therefore, $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

Thm 3.105 The annihilator is a subspace

Suppose U is a subspace of V.
Then U^0 is a subspace of V'
$(V' = \mathcal{L}(V, \mathbb{F}))$

proof:
- Additive identity: Since $0(u) = 0$ for all $u \in U$, we have $0 \in U^0$.
- Closed Under Addition: Suppose $\varphi, \psi \in U^0$. For all $u \in U$, we have $(\varphi+\psi)(u) = \varphi(u) + \psi(u)$
$= 0 + 0$
$= 0$.
So $\varphi + \psi \in U^0$
- Closed Under Scalar Multiplication: Suppose $\lambda \in \mathbb{F}$ and $\varphi \in U^0$. For all $u \in U$, we have $(\lambda\varphi)(u) = \lambda\varphi(u)$
$= \lambda \cdot 0$
$= 0$
So $\lambda\varphi \in U^0$ ☐

7/24/19

Defn 3.111 The transpose of a matrix A, denoted A^t, is the matrix obtained from A by interchanging the rows and columns.
More specifically, if

$$A = \begin{pmatrix} A_{1,1} & & A_{1,n} \\ & & \\ A_{m,1} & & A_{m,n} \end{pmatrix}, \quad \text{then} \quad A^t = \begin{pmatrix} A_{1,n} & & A_{m,1} \\ & & \\ A_{1,n} & & A_{m,n} \end{pmatrix}$$

Eg 3.112 If $A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix}$, then $A^t = \begin{pmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{pmatrix}$

Thm 3.113 The transpose of the product of matrices

Let A be an $m \times n$ matrix and B be a $n \times p$ matrix. Then
$$(AC)^t = C^t A^t$$

proof: Suppose we have $k = 1, \dots, p$ and $j = 1, \dots, m$.

Then
$$((AC)^t)_{k,j} = (AC)_{j,k}$$
$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$
$$= \sum_{r=1}^{n} (A^t)_{r,j} (C^t)_{k,r}$$
$$= \sum_{r=1}^{n} (C^t)_{k,r} (A^t)_{r,j}$$
$$= (C^t A^t)_{k,j}$$

Therefore, we have
$$(AC)^t = C^t A^t.$$ ☐

Thm 3.114 The matrix of T' is the transpose of the matrix of T

Suppose $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

proof: Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$. Suppose we have $j = 1, \dots, m$ and $k = 1, \dots, n$. →

By definition 3.32 of Axler, we have

$$Tv_k = \sum_{r=1}^{m} A_{r,k} w_r$$

and

$$T'(\varphi_j) = \sum_{r=1}^{n} C_{r,j}\varphi_r$$

So we have

$$(\varphi_j \circ T)(v_k) = (T'(\varphi_j))(v_k)$$
$$= \left(\sum_{r=1}^{n} C_{r,j}\varphi_r\right)(v_k)$$
$$= (C_{1,j}\varphi_1 + \cdots + C_{n,j}\varphi_n)(v_k)$$
$$= C_{1,j}\varphi_1(v_k) + \cdots + C_{n,j}\varphi_n(v_k)$$
$$= C_{1,j}\cancel{\varphi_1(v_k)}^{0} + \cdots + C_{k,j}\cancel{\varphi_k(v_k)}^{1} + \cdots + C_{n,j}\cancel{\varphi_n(v_k)}^{0}$$
$$= C_{1,j}\cdot 0 + \cdots + C_{k,j}\cdot 1 + \cdots + C_{n,j}\cdot 0$$
$$= C_{k,j}$$

and

$$(\psi_j \circ T)(v_k) = \psi_j(Tv_k)$$
$$= \psi_j\left(\sum_{r=1}^{n} A_{r,k} w_r\right)$$
$$= \psi_j(A_{1,k}w_1 + \cdots + A_{n,k}w_n)$$
$$= \psi_j(A_{1,k}w_1) + \cdots + \psi_j(A_{n,k}w_n)$$
$$= A_{1,k}\psi_j(w_1) + \cdots + A_{n,k}\psi_j(w_n)$$
$$= A_{1,k}\cancel{\psi_j(w_1)}^{0} + \cdots + A_{j,k}\cancel{\psi_j(w_j)}^{1} + \cdots + A_{n,k}\cancel{\psi_j(w_n)}^{0}$$
$$= A_{1,k}\cdot 0 + \cdots + A_{j,k}\cdot 0 + \cdots + A_{n,k}\cdot 0$$
$$= A_{j,k}$$

Therefore, we conclude

$$C_{k,j} = A_{j,k}.$$

So $C = A^t$, and so

$$M(T') = C = A^t = (M(T))^t$$

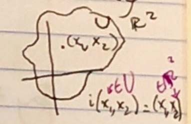
BACK TO ANNIHILATOR STUFF (3.106 – 3.110)

3.106 Dimension of the Annihilator

Suppose V is a finite-dimensional vector space and U is a subspace of V. Then

$$\dim U + \dim U^0 = \dim V.$$

proof: Let $i \in \mathcal{L}(U, V)$ be the inclusion map defined by $i(u) = u$ for all $u \in U$.



First, we will show that i' is linear.

Let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in V'$.

- Additivity: For all $u \in U$, we have

$$(i'(\varphi+\psi))(u) = ((\varphi+\psi)\circ i)(u)$$
$$= (\varphi+\psi)(i(u))$$
$$= (\varphi+\psi)(u)$$
$$= \varphi(u) + \psi(u)$$
$$= \varphi(i(u)) + \psi(i(u))$$
$$= (\varphi\circ i)(u) + (\psi\circ i)(u)$$
$$= (i'(\varphi))(u) + (i'(\psi))(u)$$
$$= (i'(\varphi) + i'(\psi))(u).$$

Therefore, $i'(\varphi+\psi) = i'(\varphi) + i'(\psi)$.

- Homogeneity: For all $u \in U$, we have

$$(i'(\lambda\varphi))(u) = ((\lambda\varphi)\circ i)(u)$$
$$= (\lambda\varphi)(i(u))$$
$$= (\lambda\varphi)(u)$$
$$= \lambda\varphi(u)$$
$$= \lambda\varphi(i(u))$$
$$= \lambda((\varphi\circ i)(u))$$
$$= (\lambda(i'(\varphi)))(u).$$

Therefore, $i'(\lambda\varphi) = \lambda i'(\varphi)$.

Hence i' is linear →

Next, we will show $\operatorname{null} i' = U^\circ$. We have

$$\begin{aligned}
\operatorname{null} i' &= \{\varphi \in V' : i'(\varphi) = 0\} \\
&= \{\varphi \in V' : \varphi \circ i = 0\} \\
&= \{\varphi \in V' : (\varphi \circ i)(u) = 0 \text{ for all } u \in U\} \\
&= \{\varphi \in V' : \varphi(i(u)) = 0 \text{ for all } u \in U\} \\
&= \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\} \\
&= U^\circ
\end{aligned}$$

By the Fundamental Theorum of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}
\dim V &= \dim V' && \text{by 3.95 of Axler} \\
&= \dim \operatorname{range} i' + \dim \operatorname{null} i' && \text{by 3.22 of Axler} \\
&= \dim \operatorname{range} i' + \dim U^\circ \\
&= \dim U' + \dim U^\circ = \dim U + \dim U^\circ
\end{aligned}$$

once we show $\operatorname{range} i' = U'$.

Show $\operatorname{range} i' = U'$.

Suppose we have $\varphi \in U'$. By Excercise 3.A.11 of Axler, we can extend to a linear functional ψ on V. And by definition of i' we have $i'(\psi) = \varphi$. So $\varphi \in \operatorname{range} i'$, and so we have $U' \subset \operatorname{range} i'$. But 3.19 of Axler says that $\operatorname{range} i'$ is a subspace of U'. Therefore, $\operatorname{range} i' = U'$, as desired. ☒

Thm 3.107 The null space of T'

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$. Then:

(a) $\operatorname{null} T' = (\operatorname{range} T)^\circ$

(b) $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$

proof: (a) Suppose $\varphi \in \operatorname{null} T'$. Then we have $T'(\varphi) = 0$.

So for all $v \in V$, we have

$$\begin{aligned}
0 = 0(v) = (T'(\varphi))(v) &= (\varphi \circ T)(v) \\
&= \varphi(Tv)
\end{aligned}$$

So $\varphi \in (\operatorname{range} T)^\circ$, and so we have

$$\operatorname{null} T' \subset (\operatorname{range} T)^\circ$$

Suppose $\varphi \in (\operatorname{range} T)^\circ$. Then $\varphi(Tv) = 0$ for all $v \in V$.

So we have

$$\begin{aligned}
(T'(\varphi))(v) &= (\varphi \circ T)(v) \\
&= \varphi(Tv) \\
&= \varphi(0) \\
&= 0
\end{aligned}$$

and so $\varphi \in \operatorname{null} T'$. So $(\operatorname{range} T)^\circ \subset \operatorname{null} T'$.

Therefore, $(\operatorname{range} T)^\circ = \operatorname{null} T$.

(b) We have

$$\begin{aligned}
\dim \operatorname{null} T' &= \dim (\operatorname{range} T)^\circ && \text{by part (a)} \\
&= \dim W - \dim \operatorname{range} T && \text{by 3.106 of Axler} \\
&= \dim W - (\dim V - \dim \operatorname{null} T) && \text{by 3.22 of Axler} \\
&= \dim \operatorname{null} T + \dim W - \dim V.
\end{aligned}$$

☒

Thm 3.108 T surjective is equivalent to T' injective.

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$. Then T is surjective if and only if T' is injective.

proof: $\Longleftrightarrow$ = "if and only if"

$$\begin{aligned}
T \in \mathcal{L}(V,W) \text{ is surjective} &\Longleftrightarrow \operatorname{range} T = W \\
&\Longleftrightarrow (\operatorname{range} T)^\circ = \{0\} \\
&\Longleftrightarrow \operatorname{null} T' = \{0\} && \text{by 3.107(a) of Axler} \\
&\Longleftrightarrow T' \text{ is injective.}
\end{aligned}$$

Prove $(\operatorname{range} T)^\circ = \{0\}$. We have

$$\begin{aligned}
\dim (\operatorname{range} T)^\circ &= \dim W - \dim \operatorname{range} T && \text{by 3.106 of Axler} \\
&= \dim W - \dim W \\
&= 0 \\
&= \dim \{0\},
\end{aligned}$$

$\Longleftrightarrow (\operatorname{range} T)^\circ = \{0\}$, by Excercise 2.C.1 of Axler.

☒

Thm 3.109 **The range of T'**

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$. Then
(a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$,
(b) $\operatorname{range} T' = (\operatorname{null} T)^\circ$.

proof: (a) We have

$$\begin{aligned}
\dim \operatorname{range} T' &= \dim W' - \dim \operatorname{null} T' && \text{by 3.22 of Axler}\\
&= \dim W - \dim \operatorname{null} T' && \text{by 3.95 of Axler}\\
&= \dim W - \dim (\operatorname{range} T)^\circ && \text{by 3.107(a) of Axler}\\
&= \dim \operatorname{range} T. && \text{by 3.106 of Axler}
\end{aligned}$$

(b) Suppose $\varphi \in \operatorname{range} T'$. Then there exists $\psi \in W'$ that satisfies $\varphi = T'(\psi)$. For all $v \in \operatorname{null} T$, we have

$$\begin{aligned}
\varphi(v) &= (T'(\psi))(v)\\
&= (\psi \circ T)(v)\\
&= \psi(Tv)\\
&= \psi(0) && \text{since } v \in \operatorname{null} T\\
&= 0.
\end{aligned}$$

So $\varphi \in (\operatorname{null} T)^\circ$. Therefore, $\operatorname{range} T' \subset (\operatorname{null} T)^\circ$.

Show $\dim \operatorname{range} T' = \dim (\operatorname{null} T)^\circ$. we have

$$\begin{aligned}
\dim \operatorname{range} T' &= \dim \operatorname{range} T && \text{by 3.109(a) of Axler}\\
&= \dim V - \dim \operatorname{null} T && \text{by Fund. Thm of Linear Maps (3.22)}\\
&= \dim (\operatorname{null} T)^\circ && \text{by 3.106 of Axler}
\end{aligned}$$

Therefore, we have in fact $\operatorname{range} T' = (\operatorname{null} T)^\circ$. ◻

Thm 3.110 **T injective is equivalent to T' surjective**

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$. Then T is injective if and only if T' is surjective.

proof:

$$\begin{aligned}
T \in \mathcal{L}(V,W) \text{ is injective} &\iff \operatorname{null} T = \{0\} && \text{by 3.16 of Axler}\\
&\iff (\operatorname{null} T)^\circ = V'\\
&\iff \operatorname{range} T' = V' && \text{by 3.109(b) of Axler}\\
&\iff T' \text{ is surjective.}
\end{aligned}$$

Prove $(\operatorname{null} T)^\circ = V'$ we have

$$\begin{aligned}
\dim (\operatorname{null} T)^\circ &= \dim V - \dim \operatorname{null} T && \text{by 3.106 of Axler}\\
&= \dim V - \dim \{0\}\\
&= \dim V - 0\\
&= \dim V\\
&= \dim V' && \text{by 3.95 of Axler,}
\end{aligned}$$

if and only if $(\operatorname{null} T)^\circ = V'$, by Exercise 2.C.1 of Axler. ◻