Therefore, by 3.56 of Axler, $\tilde{T}$ is invertible. By part a), $\tilde{T}$ is linear. Therefore, $\tilde{T}: V/(\text{null } T) \to \text{range } T$ is an isomorphism

3.F Duality

3.92 def

A linear functional on V is a linear map $\varphi: V \to \mathbb{F}$. In other words, $\varphi \in \mathcal{L}(V, \mathbb{F})$

3.93 Example

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

- Define $\varphi: \mathbb{R}^3 \to \mathbb{R}$ by
  $\varphi(x, y, z) = 4x - 5y + 2z$
  Then φ is a linear functional on $\mathbb{R}^3$
- For some $c_1, \ldots, c_n \in \mathbb{F}$, the map $\varphi: \mathbb{F}^n \to \mathbb{F}$ defined by
  $\varphi(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$
  then φ is a linear functional on $\mathbb{F}^n$
- Define $\varphi: P(\mathbb{R}) \to \mathbb{R}$ by
  $\varphi(p) = \int_0^1 p(x)\,dx$
  Then φ is a linear functional on $P(\mathbb{R})$

3.94 Def.

The dualspace of V, denoted V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbb{F})$

3.95 $\dim V' = \dim V$

Suppose V is finite-dim. Then V' is also finite-dim & $\dim V' = \dim V$

Proof:

$$\begin{aligned} \dim V' &= \dim \mathcal{L}(V, \mathbb{F}) \\ &= (\dim V)(\dim \mathbb{F}) \text{ by 3.61 of Axler} \\ &= (\dim V) \cdot 1 \\ &= \dim V \end{aligned}$$

3.96 Def

If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j for any $j = 1, \ldots, n$ is a linear functional on V that satisfies

$$\varphi_j(v_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

3.97 Example

What is the dual basis of the standard basis $e_1, \ldots, e_n$ of $\mathbb{F}^n$?

$e_1 = (1, 0, \ldots, 0)$

$e_2 = (0, 1, 0, \ldots, 0)$

$e_3 = (0,0,1,0,\ldots,0)$

$\vdots$

$e_n = (0,\ldots,0,1)$

soln: for all $j = 1,\ldots,n$ write $\varphi_j(x_1,\ldots,x_n) = c_1x_1 + \cdots + c_nx_n$

Then

$$1 = \varphi_1(v_1) = \varphi_1(e_1) = \varphi_1(1,0,\ldots,0) = c_1(1) + c_2(0) + \cdots + c_n(0) = c_1$$

$$0 = \varphi_1(v_2) = \varphi_1(e_2) = \varphi_1(0,1,0,\ldots,0) = c_1(0) + c_2(1) + c_3(0) + \cdots + c_n(0) = c$$

$\vdots$

$$0 = \varphi_1(v_n) = \varphi_1(e_n) = \varphi_3(0,\ldots,0,1) = c_1(0) + \cdots + c_{n-1}(0) + c_n(1) = c_n$$

So

$$\varphi_1(x_1,\ldots,x_n) = \overset{c_1}{1}x_1 + \overset{c_2}{0}x_2 + \cdots + \overset{c_n}{0}x_n = x_1$$

Similarly,

$\varphi_1(x_1,\ldots,x_n) = x_1$

$\varphi_2(x_1,\ldots,x_n) = x_2$

$\varphi_3(x_1,\ldots,x_n) = x_3$

$\vdots$

$\varphi_n(x_1,\ldots,x_n) = x_n$

So $\varphi_1,\ldots,\varphi_n$ as defined above is the dual basis of $v_1,\ldots,v_n$

Axler

Define φ_j to be the linear functional on $\mathbb{F}^n$ that selects the j^{th} coordinate of a vector in $\mathbb{F}^n$

In other words, $\varphi_j(x_1,\ldots,x_n) = x_j$ for all $(x_1,\ldots,x_n) \in \mathbb{F}^n$

3.98 Dual basis is a basis of the dual space

Suppose V is finite-dim. Then the dual basis of a basis of V is a basis of V'

Proof:

Let $v_1,\ldots,v_n$ be a basis of V, & let $\varphi_1,\ldots,\varphi_n$ be a dual basis of $v_1,\ldots,v_n$. We will show that $\varphi_1,\ldots,\varphi_n$ is lin indep. Suppose $a_1,\ldots,a_n \in \mathbb{F}$ satisfy $a_1\varphi_1 + \cdots + a_n\varphi_n = 0$

Then for any $j = 1,\ldots,n$ we have

$$\begin{aligned}(a_1\varphi_1 + \cdots + a_n\varphi_n)(v_j) &= a_1\varphi_1(v_j) + \cdots + a_n\varphi_n \\ &= a_1\varphi_1(v_j) + \cdots + a_j\varphi_j(v_j) + \cdots + a_n\varphi_n(\\ &= a_1 \cdot 0 + \cdots + a_j 1 + \cdots + a_n \cdot 0 \\ &= a_j\end{aligned}$$

so we have

$$\begin{aligned}a_j &= (a_1\varphi_1 + \cdots + a_n\varphi_n)(v_j) \\ &= 0(v_j) \\ &= 0\end{aligned}$$

In other words,

$a_1 = 0, \dots, a_n = 0$

So $\varphi_1, \dots, \varphi_n$ is lin indep.

By 3.95 of Axler, $\dim V' = \dim V$

By 2.39 of Axler, $\varphi_1, \dots, \varphi_n$ is a basis of V'

3.99 Def

If $T \in \mathcal{L}(V, W)$, then the <u>dual map</u> of T is the linear map $T' \in \mathcal{L}(W', V')$ defined by

$$T'(\varphi) = \varphi \circ T$$

end of def.

show: $T' \in \mathcal{L}(W', V')$

Let $\lambda \in \mathbb{F}$ & $\varphi, \psi \in W'$ be arbitrary

- additivity: $T'(\varphi + \psi) = (\varphi + \psi) \circ T$
$= \varphi \circ T + \psi \circ T$
$= T'(\varphi) + T'(\psi)$
- Homogeneity: $T'(\lambda\varphi) = (\lambda\varphi) \circ T$
$= \lambda(\varphi \circ T)$
$= \lambda T'(\varphi)$

3.100 Example

Define $D: P(\mathbb{R}) \to P(\mathbb{R})$ by $Dp = p'$

- define $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ by $\varphi(p) = p(3)$

Then $D'(\varphi)$ is the linear functional on $P(\mathbb{R})$ that satisfies

$$(D'(\varphi))(p) = (\varphi \circ D)(p)$$
$$= \varphi(Dp)$$
$$= \varphi(p')$$
$$= p'(3)$$

- Define $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ by

$$\varphi(p) = \int_0^1 p(x)\,dx$$

Then $D'(\varphi)$ is the linear functional on $P(\mathbb{R})$ given by

$$(D'(\varphi))(p) = (\varphi \circ D)(p)$$
$$= \varphi(Dp)$$
$$= \varphi(p')$$
$$= \int_0^1 p'(x)\,dx$$
$$= p(1) - p(0)$$

3.101 Algebraic Properties of Dual Maps

a) $(S+T)' = S' + T'$ for all $S, T \in \mathcal{L}(V, W)$

b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$ & for all $T \in \mathcal{L}(V, W)$

c) $(ST)' = T'S'$ for all $T \in \mathcal{L}(U, V)$ & for all $S \in \mathcal{L}(V, W)$

proof:

a) For all $\varphi \in \mathcal{L}(V, \mathbb{F})$ we have

$(S+T)'(\varphi) = \varphi \circ (S+T)$
$= \varphi \circ S + \varphi \circ T$
$= S'(\varphi) + T'(\varphi)$

b) For all $\varphi \in \mathcal{L}(V, \mathbb{F})$

$(\lambda T)'(\varphi) = \varphi \circ (\lambda T)$
$= \lambda \varphi \circ T$
$= \lambda T'(\varphi)$

c) : for all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$(ST)'(\varphi) = \varphi \circ (ST)$
$= (\varphi \circ S) \circ T$
$= T'(\varphi \circ S)$
$= T'(S'(\varphi))$
$= (T'S')(\varphi)$
so $(ST)'(\varphi) = T'S'$

3.102 Def

If U is a subspace of V, then the annihilator of U, denoted U^0, is defined

$U^0 = \{\varphi \in V' : \varphi(u) = \emptyset \text{ for all } u \in U\}$

3.103 example

Let U be the subspace of $P(\mathbb{R})$ consisting of all polynomial multiples of x^2, such as $x^2 p(x)$

If $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ is defined by $\varphi(p) = p'(0)$, then $\varphi \in U^0$

$x^2 p(x) \in U$, if $p \in P(\mathbb{R})$

and $\varphi(x^2 p(x)) = (x^2 p(x))' \big|_{x=0}$
$= (2x\,p(x) + x^2 p'(x)) \big|_{x=0}$
$= 2(0)p(0) + 0^2 p'(0)$
$= \emptyset$

3.104 example

Let e_1, e_2, e_3, e_4, e_5 denote the standard basis of $\mathbb{R}^5$, ';

Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ denote the dual basis of $(\mathbb{R}^5)'$

Suppose

$U = \operatorname{span}(e_1, e_2)$
$= \{(x_1, x_2, 0, 0, 0) \in \mathbb{R}^5 : x_1, x_2 \in \mathbb{R}\}$

show $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

soln: Recall from example 3.97 that

$\varphi_j(x_1, x_2, x_3, x_4, x_5) = x_j$

for any $j = 1, 2, 3, 4, 5$

Suppose we have $\varphi \in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

Then

$\varphi = c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$

for some $c_3, c_4, c_5 \in \mathbb{R}$. for all $(x_1, x_2, 0, 0, 0) \in U$, we have

$$\begin{aligned}\varphi(x_1, x_2, 0, 0, 0) &= (c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(x_1, x_2, 0, 0, 0) \\ &= c_3\varphi_3(x_1, x_2, 0, 0, 0) + c_4\varphi_4(x_1, x_2, 0, 0, 0) + c_5\varphi_5(x_1, x_2, 0, 0, 0) \\ &= c_3 0 + c_4 0 + c_5 0 \\ &= \emptyset\end{aligned}$$

So $\varphi \in U^0$, so $\operatorname{span}(\varphi_3, \varphi_4, \varphi_5) \subset U^0$

Suppose $\varphi \in U^0$. Since $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ is the dual basis of $(\mathbb{R}^5)'$, we can write uniquely as $\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$ for some $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$.

Since $e_1 \in U$, we have

$$\begin{aligned}\emptyset &= \varphi(e_1) \\ &= (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_1) \\ &= c_1\varphi_1(e_1) + c_2\varphi(e_1) + c_3\varphi_3(e_1) + c_4\varphi_4(e_1) + c_5\varphi_5(e_1) \\ &= c_1 1 + c_2 0 + c_3 0 + c_4 0 + c_5 0 \\ &= c_1\end{aligned}$$

Since $e_2 \in U$

$$\begin{aligned}0 &= \varphi(e_2) \\ &= (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_2) \\ &= c_1\varphi_1(e_2) + c_2\varphi_2(e_2) + c_3\varphi_3(e_2) + c_4\varphi_4(e_2) + c_5\varphi_5(e_2) \\ &= c_1 0 + c_2 1 + c_3 0 + c_4 0 + c_5 0 \\ &= c_2\end{aligned}$$

Therefore,
$$\begin{aligned}\varphi &= c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \\ &= 0\varphi_1 + 0\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \\ &= c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \\ &\in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)\end{aligned}$$

So $U^0 \subset \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

Therefore, $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

<u>3.105 The annihilator is a subspace</u>

Suppose U is a subspace of V

Then U^0 is a subspace of V'

$(V' = \mathcal{L}(V, \mathbb{F}))$

<u>Proof:</u>

- Add Id: since $0(u) = 0$ for all $u \in U$, we have $0 \in U^0$
- closed under add: Suppose $\varphi, \psi \in U^0$ for all $u \in U$, we have

$$\begin{aligned}(\varphi + \psi)(u) &= \varphi(u) + \psi(u) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

So $\varphi + \psi \in U^0$

°closed under scalar mult

Suppose $\lambda \in F$ & $\varphi \in U^0$ for all $u \in U$

we have

$$(\lambda\varphi)(u) = \lambda\varphi(u)$$
$$= \lambda \cdot 0$$
$$= \emptyset$$

7/23/19 so $\lambda\varphi \in U^0$

Wed **3.111 Def**

week 5 The transpose of a matrix A, denoted A^t, is the matrix obtained from A by interchanging the rows & columns. More specifically, if

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}, \text{ then } A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{pmatrix}$$

3.112 Example

If $A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix}$, then $A^t = \begin{pmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{pmatrix}$

3.113 The transpose of the product of matrices

Let A be an $m \times n$ matrix & B be an $n \times p$ matrix. Then

$$(AC)^t = C^t A^t$$

Proof: Suppose we have

$k = 1, \ldots, p$ & $j = 1, \ldots, m$

Then
$$((AC)^t)_{k,j} = (AC)_{j,k}$$
$$= \sum_{r=1}^{n} A_{j,r} C_{r,k}$$
$$= \sum_{r=1}^{n} (A^t)_{r,j} (C^t)_{k,r}$$
$$= \sum_{r=1}^{n} (C^t)_{k,r} (A^t)_{r,j}$$
$$= (C^t A^t)_{k,j}$$

Therefore, we have

$(AC)^t = C^t A^t$. ~end of proof~

3.114 The matrix of T' is the transpose of the matrix of T

Suppose $T \in \mathcal{L}(V, W)$. Then $M(T') = (M(T))^t$

proof:

Let $A = M(T)$ & $C = M(T')$

Suppose we have $j = 1, \ldots, m$ & $k = 1, \ldots, n$.

By def 3.32 of Axler, we have

$$Tv_k = \sum_{r=1}^{m} A_{r,k} w_r$$

and

$$T'(\psi_j) = \sum_{r=1}^{n} c_{r,j} \varphi_r$$

So we have

$$(\psi_j \circ T)(v_k) = (T'(\psi_j))(v_k)$$
$$= \left(\sum_{r=1}^{n} c_{r,j} \varphi_r\right)(v_k)$$
$$= (c_{1,j}\varphi_1 + \dots + c_{n,j}\varphi_n)(v_k)$$
$$= c_{1,j}\varphi_1(v_k) + \dots + c_{n,j}\varphi_n(v_k)$$
$$= c_{1,j}\cancelto{0}{\varphi_1(v_k)} + \dots + c_{k,j}\cancelto{1}{\varphi_k(v_k)} + \dots + c_{n,j}\varphi_n(v_k)$$
$$= c_{1,j}\cdot 0 + \dots + c_{k,j}\cdot 1 + \dots + c_{n,j}\cdot 0$$
$$= c_{k,j}$$

and

$$(\psi_j \circ T)(v_k) = \psi_j(Tv_k)$$
$$= \psi_j\left(\sum_{r=1}^{n} A_{r,k} w_r\right)$$
$$= \psi_j(A_{1,k}w_1 + \dots + A_{n,k}w_n)$$
$$= \psi_j(A_{1,k}w_1) + \dots + \psi_j(A_{m,k}w_n)$$
$$= A_{1,k}\psi_j(w_1) + \dots + A_{n,k}\psi_j(w_n)$$
$$= A_{1,k}\cancelto{0}{\psi_j(w_1)} + \dots + A_{j,k}\cancelto{1}{\psi_j(w_j)} + \dots + A_{n,k}\cancelto{0}{\psi_j(w_n)}$$
$$= A_{1,k}\cdot 0 + \dots + A_{j,k}\cdot 1 + \dots + A_{n,k}\cdot 0$$
$$= A_{j,k}$$

Therefore, we conclude

$$c_{k,j} = A_{j,k} \text{ so } C = A^t, \text{ \& so } M(T') = C = A^t = (M(T))^t$$

~ end proof ~

BACK TO ANNIHILATOR STUFF
(3.106 – 3.110)

3.106 Dimension of the annihilator

Suppose V is a finite-dim vector space & U is a subspace of V. Then $\dim U + \dim U^0 = \dim V$.

Proof:

Let $i \in \mathcal{L}(U, V)$ be the inclusion map defined by $i(u) = u$ for all $u \in U$.

First we will show that i' is linear.

Let $\lambda \in \mathbb{F}$ & $\varphi, \psi \in V'$

• Additivity: For all $u \in U$, we have

$$\begin{aligned}(i'(\varphi+\psi))(u) &= ((\varphi+\psi)\circ i)(u)\\ &= (\varphi+\psi)(i(u))\\ &= (\varphi+\psi)(u)\\ &= \varphi(u)+\psi(u)\\ &= \varphi(i(u))+\psi(i(u))\\ &= (\varphi\circ i)(u)+(\psi\circ i)(u)\\ &= (i'(\varphi)(u)+(i'(\psi))(u)\\ &= (i'(\varphi)+i'(\psi))(u)\end{aligned}$$

Therefore, $i'(\varphi+\psi) = i'(\varphi)+i'(\psi)$

• Homogeneity: for all $u \in U$, we have

$$\begin{aligned}(i'(\lambda\varphi))(u) &= ((\lambda\varphi)\circ i)(u)\\ &= (\lambda\varphi)(i(u))\\ &= (\lambda\varphi)(u)\\ &= \lambda\varphi(u)\\ &= \lambda\varphi(i(u))\\ &= \lambda((\varphi\circ i)(u))\\ &= (\lambda(i'(\varphi))(u)\end{aligned}$$

Therefore, $i'(\lambda\varphi) = \lambda i'(\varphi)$

Next we will show null $i' = U^0$

we have

$$\begin{aligned}\text{null } i' &= \{\varphi\in V' : i'(\varphi)=\emptyset\}\\ &= \{\varphi\in V' : \varphi\circ i=\emptyset\}\\ &= \{\varphi\in V' : (\varphi\circ i)(u)=\emptyset \text{ for all } u\in U\}\\ &= \{\varphi\in V' : \varphi(i(u))=\emptyset \text{ for all } u\in U\}\\ &= \{\varphi\in V' : \varphi(u)=\emptyset \text{ for all } u\in U\}\\ &= U^0\end{aligned}$$

By the fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim V &= \dim V' \quad \text{by 3.95 of Axler}\\ &= \dim \text{range } i' + \dim \text{null } i' \quad \text{Fund Thm of Linear Map (3.22 of Axler)}\\ &= \dim \text{range } i' + \dim U^0\\ &= \dim U' + \dim U^0 \quad = \dim U + \dim U^0\end{aligned}$$

once we show range $i' = U$

<u>Show range $i' = U'$</u>

Suppose we have $\varphi \in U'$. By exercise 3.A.11 of Axler, we can extend to a linear functional ψ on V & by def of i' we have $i'(\psi) = \varphi$. So $\varphi \in$ range i', & so we have $U \subset$ range i'. But 3.19 of Axler says that range i' is a subspace of U'. Therefore, range $i' = U'$, as desired

~ end proof ~

3.107 The nullspace of T'

Suppose V & W are finite-dim vector spaces & $T \in \mathcal{L}(V, W)$. Then:

a) null $T' = (\text{range } T)^0$

b) dim null $T' = \dim \text{null } T + \dim W - \dim V$,

Proof:

a) Suppose $\varphi \in \text{null } T'$. Then we have $T'(\varphi) = \emptyset$ so for all $v \in V$, we have

$$\begin{aligned} 0 &= 0(v) \\ &= (T'(\varphi))(v) \\ &= (\varphi \circ T)(v) \\ &= \varphi(Tv) \end{aligned}$$

so $\varphi \in (\text{range } T)^0$, & so we have null $T' \subset (\text{range } T)^0$

Suppose $\varphi \in (\text{range } T)^0$. Then $\varphi(Tv) = \emptyset$ for all $v \in V$.

So we have

$$\begin{aligned} (T'(\varphi))(v) &= (\varphi \circ T)(v) \\ &= \varphi(Tv) \\ &= \varphi(0) \\ &= 0 \end{aligned}$$

& so $\varphi \in \text{null } T'$. So $(\text{range } T)^0 \subset \text{null } T'$

Therefore, $(\text{range } T)^0 = \text{null } T'$

b) We have

$$\begin{aligned} \dim \text{null } T' &= \dim (\text{range } T)^0 \quad \text{by part a)} \\ &= \dim W - \dim \text{range } T \quad \text{by 3.106 of Axler} \\ &= \dim W - (\dim V - \dim \text{null } T) \end{aligned}$$

by fund. Thm. of linear maps (3.22 of Axler)

$$= \dim \text{null } T + \dim W - \dim V$$

~ end proof ~

3.108 T surjective is equivalent to T' injective

Suppose V & W are finite-dim vector spaces & $T \in \mathcal{L}(V, W)$. Then T is surjective if & only if T' is injective

Proof: $\Longleftrightarrow$ = "if & only if"

$$\begin{aligned} T \in \mathcal{L}(V, W) \text{ is surjective} &\Longleftrightarrow \text{range } T = W \\ &\Longleftrightarrow (\text{range } T)^0 = \{0\} \\ &\Longleftrightarrow \text{null } T' = \{0\} \quad \text{by 3.107 a) of Axler} \\ &\Longleftrightarrow T' \text{ is injective} \end{aligned}$$

Prove $(\text{range } T)^0 = \{0\}$

we have

$\dim (\text{range } T)^0 = \dim W - \dim \text{range } T$ by 3.106 of Axler

$= \dim W - \dim W$

$= \emptyset$

$= \dim \{0\}$,

iff 's only if $(\text{range } T)^0 = \{0\}$, by exercise 2.C.1 of Axler.

($\Leftarrow\Rightarrow$)

3.109 The range of T'

Suppose V 's W are finite-dim vector spaces 's $T \in \mathcal{L}(V, W)$. Then

a) $\dim \text{range } T' = \dim \text{range } T$,

b) $\text{range } T' = (\text{null } T)^0$

proof:

a) we have

$\dim \text{range } T' = \dim W' - \dim \text{null } T'$ by the fund thm of linear maps (3.22 of Axler

$= \dim W - \dim \text{null } T'$ by 3.95

$= \dim W - \dim (\text{range } T)^0$ by 3.107 a)

$= \dim \text{range } T$ by 3.106

b) Suppose $\varphi \in \text{range } T'$. Then there exists $\psi \in W'$ that satisfies $\varphi = T'(\psi)$ for all $v \in \text{null } T$, we have $\varphi(v) = (T'(\psi))(v)$

$= (\psi \circ T)(v)$

$= \psi(Tv)$

$= \psi(\emptyset)$ since $v \in \text{null } T$

$= \emptyset$ so $\varphi \in (\text{null } T)^0$.

Therefore, $\text{range } T' \subset (\text{null } T)^0$

Show $\dim \text{range } T' = \dim (\text{null } T)^0$

we have

$\dim \text{range } T' = \dim \text{range } T$ by 3.109 a) of Axler

$= \dim V - \dim \text{null } T$ by fund thm of linear maps (3.22 of Axler

$= \dim (\text{null } T)^0$ by 3.106 of Axler

Therefore, we have in fact $\text{range } T' = (\text{null } T)^0$

~ end proof ~

3.110 T injective is equivalent to T' surjective

Suppose V 's W are finite-dim vector spaces 's $T \in \mathcal{L}(V, W)$. Then T is injective if 's only if T' is surjective

Proof: $T \in \mathcal{L}(V, W)$ is injective $\Leftarrow\Rightarrow \text{null } T = \{0\}$ by 3.16 of Axler

$\Leftarrow\Rightarrow (\text{null } T)^0 = V'$

$\Leftarrow\Rightarrow \text{range } T' = V'$ by 3.109 b) of Axler

$\Leftarrow\Rightarrow T'$ is surjective

Prove $(\text{null } T)^0 = V'$

we have $\dim (\text{null } T)^0 = \dim V - \dim \text{null } T$ by 3.106 of Axler

$= \dim V - \dim \{0\}$

$= \dim V - \emptyset$

$= \dim V,$

$= \dim V'$ by 3.95 of Axler,

if & only if $(\text{null}\, T)^0 = V$, by exercise 2.C.1 of Axler