07-23-19

3.F Duality

3.92 Definition

A linear functional on V is a linear map $\varphi: V \to \mathbb{F}$
In other words, $\varphi \in \mathcal{L}(V, \mathbb{F})$

3.93 Example

- Define $\varphi: \mathbb{R}^3 \to \mathbb{R}$ by $\quad \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$$\varphi(x, y, z) = 4x - 5y + 2z$$

Then φ is a linear functional on $\mathbb{R}^3$

- For some $c_1, \dots, c_n \in \mathbb{F}$, the map $\varphi: \mathbb{F}^n \to \mathbb{F}$ defined by

$$\varphi(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

then φ is a linear functional on $\mathbb{F}^n$

- Define $\varphi: P(\mathbb{R}) \to \mathbb{R}$ by

$$\varphi(p) = \int_0^1 p(x)\,dx$$

Then φ is a linear functional on $P(\mathbb{R})$

3.94 Definition

The dual space of V, denoted V', is the vector space of all linear function's on V. In other words

$$V' = \mathcal{L}(V, \mathbb{F})$$

3.95 $\dim V' = \dim V$

Suppose V is finite-dimensional. Then V' is also finite-dimensional and

$$\dim V' = \dim V$$

Proof: $\dim V' = \dim \mathcal{L}(V, \mathbb{F})$
$= (\dim V)(\dim \mathbb{F})$ by 3.61 of Axler
$= (\dim V) \cdot 1$
$= \dim V$ ▨

3.96 Definition

If $v_1, \dots, v_n$ is a basis of V, then the dual basis of $v_1, \dots, v_n$ is the list $\varphi_1, \dots, \varphi_n$ of elements of V', where each φ_j for any $j = 1, \dots, n$ is a linear functional on V that satisfies

$$\varphi_j(v_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

3.97 Example

what is the dual basis of the standard basis $e_1, \dots, e_n$ of $\mathbb{F}^n$?

$e_1 = (1, 0, \dots, 0)$

$e_0 = (0, 1, 0, \dots, 0)$

$e_2 = (0, 0, 1, 0, \dots, 0)$

$\vdots$

$e_n = (0, \dots, 0, 1)$

Solution: For all $j = 1, \dots, n$, write

$\varphi_j(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$

Then

$1 = \varphi_1(v_1) = \varphi_1(e_1) = \varphi_1(1, 0, \dots, 0) = c_1(1) + c_2(0) + \dots + c_n(0) = c_1$

$0 = \varphi_1(v_2) = \varphi_1(e_2) = \varphi_1(0, 1, 0, \dots, 0) = c_1(0) + c_2(1) + c_3(0) + \dots + c_n(0) = c_2$

$\vdots$

$0 = \varphi_1(v_n) = \varphi_1(e_n) = \varphi_3(0, \dots, 0, 1) = c_1(0) + \dots + c_{n-1}(0) + c_n(1) = c_n$

So

$\varphi_1(x_1, \dots, x_n) = \underset{c_1}{(1)} x_1 + \underset{c_2}{(0)} x_2 + \dots + \underset{c_n}{(0)} x_n$

$= x_1$

$c_1 = 1$

$c_2 = 0$

$\vdots$

$c_n = 0$

Similarly

$\varphi_1(x_1, \dots, x_n) = x_1$

$\varphi_2(x_1, \dots, x_n) = x_2$

$\varphi_3(x_1, \dots, x_n) = x_3$

$\vdots$

$\varphi_n(x_1, \dots, x_n) = x_n$

So $\varphi_1, \dots, \varphi_n$ as defined above is the dual basis of $v_1, \dots, v_n$

Axler

Define φ_j to be the linear functional on $\mathbb{F}^n$ that selects the j^{th} coordinate of a vector in $\mathbb{F}^n$

In other words,

$\varphi_j(x_1, \dots, x_n) = x_j$

for all $(x_1, \dots, x_n) \in \mathbb{F}^n$

3.98 Dual basis is a basis of the dual space

Suppose V is finite-dimensional.

Then the dual basis of a basis of V is a basis of V'

Proof: Let $v_1, \dots, v_n$ be a basis of V,

and let $\varphi_1, \dots, \varphi_n$ be a dual basis of $v_1, \dots, v_n$.

we will show that $\varphi_1, \dots, \varphi_n$ is linearly independent

Suppose $a_1, \dots, a_n \in \mathbb{F}$ satisfy

$a_1\varphi_1 + \dots + a_n\varphi_n = 0$

Then, for any $j = 1, \dots, n$ we have

$$(a_1\varphi_1 + \cdots + a_n\varphi_n)(v_j) = a_1\varphi_1(v_j) + \cdots + a_n\varphi_n(v_j)$$
$$= a_1\varphi_1(v_j) + \cdots + a_j\varphi_j(v_j) + \cdots + a_n\varphi_n(v_j)$$
$$= a_1 \cdot 0 + \cdots + a_j \cdot 1 + \cdots + a_n \cdot 0$$
$$= a_j$$

So we have

$$a_j = (a_1\varphi_1 + \cdots + a_n\varphi_n)(v_j)$$
$$= 0 \cdot (v_j)$$
$$= 0$$

In other words

$$a_1 = 0, \dots, a_n = 0$$

So $\varphi_1, \dots, \varphi_n$ is linearly independent

By 3.95 of Axler, $\dim V' = \dim V$

By ~~[illegible]~~ 3.39 of Axler, $\varphi_1, \dots, \varphi_n$ is a basis of V' $\blacksquare$

3.99 Definition

If $T \in \mathcal{L}(V, W)$ then the dual map of T is the linear map $T' \in \mathcal{L}(W', V')$ defined by

$$T'(\underbrace{\varphi}_{\text{input of linear functional}}) = \underbrace{\varphi \circ T}_{\text{output of linear functional}}$$

Show: $T' \in \mathcal{L}(W', V')$

Let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in W'$ be arbitrary

- Additivity: $T'(\varphi + \psi) = (\varphi + \psi) \circ T$
$$= \varphi \circ T + \psi \circ T$$
$$= T'(\varphi) + T'(\psi)$$
- Homogeneity: $T'(\lambda\varphi) = (\lambda\varphi) \circ T$
$$= \lambda(\varphi \circ T)$$
$$= \lambda T'(\varphi)$$

3.100 Example

Define $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $Dp = p'$

- Define $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ by
$$\varphi(p) = p(3)$$

Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ that satisfies

$$(D'(\varphi))(p) = (\varphi \circ D)(p)$$
$$= \varphi(Dp)$$
$$= \varphi(p')$$
$$= p'(3)$$

• Define $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ by

$$\varphi(p) = \int_0^1 p(x)\,dx$$

then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ given by

$$\begin{aligned}(D'(\varphi))(p) &= (\varphi \circ D)(p) \\ &= \varphi(Dp) \\ &= \varphi(p') \\ &= \int_0^1 p'(x)\,dx \\ &= p(1) - p(0)\end{aligned}$$

3.101 Algebraic properties of Dual Maps

(a) $(S+T)' = S' + T'$ for all $S, T \in \mathcal{L}(V, W)$

(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$ and for all $T \in \mathcal{L}(V, W)$

(c) $(ST)' = T'S'$ for all $T \in \mathcal{L}(U, V)$ and for all $S \in \mathcal{L}(V, W)$

Proof: (a) For all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$\begin{aligned}(S+T)'(\varphi) &= \varphi \circ (S+T) \\ &= \varphi \circ S + \varphi \circ T \\ &= S'(\varphi) + T'(\varphi)\end{aligned}$$

(b) For all $\varphi \in \mathcal{L}(V, \mathbb{F})$,

$$\begin{aligned}(\lambda T)'(\varphi) &= \varphi \circ (\lambda T) \\ &= \lambda \varphi \circ T \\ &= \lambda T'(\varphi)\end{aligned}$$

(c) For all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$\begin{aligned}(ST)'(\varphi) &= \varphi \circ (ST) \\ &= (\varphi \circ S) \circ T \\ &= T'(\varphi \circ S) \\ &= T'(S'(\varphi)) \\ &= (T'S')(\varphi)\end{aligned}$$

So $(ST)'(\varphi) = T'S'$

3.102 Definition

If U is a subspace of V, then the annihilator of U, denoted U^0, is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$$

3.103 Example

Let U be the subspace of $P(\mathbb{R})$ consisting of all polynomials multiples of x^2 such as $x^2 p(x)$.

If $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ is defined by

$$\varphi(p) = p'(0),$$

then $\varphi \in U^0$.

$x^2 p(x) \in U$. if $p \in P(\mathbb{R})$

And

$$\varphi(x^2 p(x)) = (x^2 p(x))' |_{x=0}$$
$$= (2x\, p(x) + x^2 p'(x)) |_{x=0}$$
$$= 2(0)\, p(0) + 0^2 p'(0) = 0$$

3.104 Example

Let e_1, e_2, e_3, e_4, e_5 denote the standard basis of $\mathbb{R}^5$ and let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ denote the dual basis of $(\mathbb{R}^5)'$

Suppose

$$U = \operatorname{span}(e_1, e_2) = \{ (x_1, x_2, 0, 0, 0) \in \mathbb{R}^5 : x_1, x_2 \in \mathbb{R} \}$$

Show $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

Solution: Recall from Example 3.97 that

$$\varphi_j(x_1, x_2, x_3, x_4, x_5) = x_j$$

for any $j = 1, 2, 3, 4, 5$

Suppose we have $\varphi \in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

Then $\varphi = c_3 \varphi_3 + c_4 \varphi_4 + c_5 \varphi_5$

for some $c_3, c_4, c_5 \in \mathbb{R}$. For all $(x_1, x_2, 0, 0, 0) \in U$

we have

$$\varphi(x_1, x_2, 0, 0, 0) = (c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(x_1, x_2, 0, 0, 0)$$
$$= c_3\varphi_3(x_1, x_2, 0, 0, 0) + c_4\varphi_4(x_1, x_2, 0, 0, 0) + c_5\varphi_5(x_1, x_2, 0, 0, 0)$$
$$= c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0$$
$$= 0$$

Suppose $\varphi \in U^0$. Since $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ is the dual basis of $(\mathbb{R}^5)'$, we can write uniquely as

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$$

for some $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$

Since $e_1 \in U$, we have

$$0 = \varphi(e_1)$$
$$= (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_1)$$
$$= c_1\varphi_1(e_1) + c_2\varphi_2(e_1) + c_3\varphi_3(e_1) + c_4\varphi_4(e_1) + c_5\varphi_5(e_1)$$
$$= c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0$$
$$= c_1$$

Since $e_2 \in U$,

$$0 = \varphi(e_2)$$
$$= (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_2)$$
$$= c_1\varphi_1(e_2) + c_2\varphi_2(e_2) + c_3\varphi(e_2) + c_4\varphi_4(e_2) + c_5\varphi_5(e_2)$$
$$= c_1 \cdot 0 + c_2 \cdot 1 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0$$
$$= c_2$$

Therefore,

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 = 0\varphi_1 + 0\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$$
$$= c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$$

so $U^0 \subset \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

Therefore $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

3.105 The annihilator is a subspace

Suppose U is a subspace of V. Then U^0 is a subspace of V' $(V' = \mathcal{L}(V, \mathbb{F}))$

Proof:

- Additivity: Since $0(u) = 0$ for all $u \in U$, we have $0 \in U^0$
- Closed under addition: Suppose $\varphi, \psi \in U^0$. For all $u \in U$, we have

$$(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0 + 0 = 0$$

So $\varphi + \psi \in U^0$

- closed under scalar multiplication:

Suppose $\lambda \in \mathbb{F}$ and $\varphi \in U^0$. For $u \in U$ we have

$$(\lambda\varphi)(u) = \lambda\varphi(u) = \lambda \cdot 0 = 0$$

So $\lambda\varphi \in U^0$

07-24-19

3.111 Definition

The transpose of a matrix A, denoted A^t, is the matrix obtained from A by interchanging the rows and columns.

More specifically, if

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}, \quad \text{then} \quad A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{pmatrix}$$

3.112 Example

If $A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix}$, then $A^t = \begin{pmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{pmatrix}$

3.113 The transpose of the product of matrices

Let A be a $m \times n$ matrix and B be an $n \times p$ matrix. Then

$$(AC)^t = C^t A^t$$

Proof: Suppose we have $k = 1, \ldots, p$ and $j = 1, \ldots, m$

Then
$$\begin{aligned}((AC)^t)_{k,j} &= (AC)_{j,k} \\ &= \sum_{r=1}^{n} A_{j,r} C_{r,k} \\ &= \sum_{r=1}^{n} (A^t)_{r,j} (C^t)_{k,r} \\ &= \sum_{r=1}^{n} (C^t)_{k,r} (A^t)_{r,j} \\ &= (C^t A^t)_{k,j}\end{aligned}$$

Therefore we have

$$(AC)^t = C^t A^t$$

3.114 The matrix of T' is the transpose of the matrix of T

Suppose $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(T') = (\mathcal{M}(T))^t$

Proof: Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$

Suppose we have $j = 1, \dots, m$ and $k = 1, \dots, n$.

By Definition 3.32 of Axler, we have

$$Tv_k = \sum_{r=1}^{m} A_{r,k} w_r$$

and

$$T'(\psi_j) = \sum_{r=1}^{n} C_{r,j} \varphi_r$$

so we have

$$\begin{aligned}
(\psi_j \circ T)(v_k) &= (T'(\varphi_{rj}))(v_k) \\
&= \left(\sum_{r=1}^{n} C_{r,j} \varphi_r\right)(v_k) \\
&= (C_{1,j}\varphi_1 + \dots + C_{n,j}\varphi_n)(v_k) \\
&= C_{1,j}\varphi_1(v_k) + \dots + C_{n,j}\varphi_n(v_k) \\
&= C_{1,j}\varphi_1(v_k) + \dots + C_{k,j}\varphi_k(v_k) + \dots + C_{n,j}\varphi_n(v_k) \\
&= C_{1,j} \cdot 0 + \dots + C_{k,j} \cdot 1 + \dots + C_{n,j} \cdot 0 \\
&= C_{k,j}
\end{aligned}$$

and

$$\begin{aligned}
(\psi_j \circ T)(v_k) &= \psi_j(Tv_k) \\
&= \psi_j\left(\sum_{r=1}^{n} A_{r,k} w_r\right) \\
&= \psi_j(A_{1,k}w_1 + \dots + A_{n,k}w_n) \\
&= \psi_j(A_{1,k}w_1) + \dots + \psi_j(A_{n,k}w_n) \\
&= A_{1,k}\psi_j(w_1) + \dots + A_{n,k}\psi_j(w_n) \\
&= A_{1,k}\psi_j(w_1) + \dots + A_{j,k}\psi_j(w_j) + \dots + A_{n,k}\psi_j(w_n) \\
&= A_{1,k} \cdot 0 + \dots + A_{j,k} \cdot 1 + \dots + A_{n,k} \cdot 0 \\
&= A_{j,k}
\end{aligned}$$

Therefore we conclude

$$C_{k,j} = A_{j,k}$$

so

$$C = A^t,$$

and so

$$\begin{aligned}
\mathcal{M}(T') &= C \\
&= A^t \\
&= (\mathcal{M}(T))^t
\end{aligned}$$

∎

BACK TO ANNIHILATOR STUFF (3.106 - 3.110)

3.106 Dimension of the annihilator

Suppose V is a finite-dimensional vector space and U is a subspace of V. Then

$$\dim U + \dim U^0 = \dim V.$$

Proof: Let $i \in \mathcal{L}(U, V)$ be the inclusion map defined by $i(u) = u$ for all $u \in U$

First, we will show that i' is linear

Let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in V'$

- Additivity: For all $u \in U$ we have

$$\begin{aligned}
(i'(\varphi + \psi))(u) &= ((\varphi + \psi) \circ i)(u) \\
&= (\varphi + \psi)(i(u)) \\
&= (\varphi + \psi)(u) \\
&= \varphi(u) + \psi(u) \\
&= \varphi(i(u)) + \psi(i(u)) \\
&= (\varphi \circ i)(u) + (\psi \circ i)(u) \\
&= (i'(\varphi))(u) + (i'(\psi))(u) \\
&= (i'(\varphi) + i'(\psi))(u)
\end{aligned}$$

Therefore $i'(\varphi + \psi) = i'(\varphi) + i'(\psi)$

- Homogeneity: Fore all $u \in U$, we have

$$\begin{aligned}
(i'(\lambda\varphi))(u) &= ((\lambda\varphi) \circ i)(u) \\
&= (\lambda\varphi)(i(u)) \\
&= (\lambda\varphi)(u) \\
&= \lambda\varphi(u) \\
&= \lambda\varphi(i(u)) \\
&= \lambda((\varphi \circ i)(u)) \\
&= (\lambda(i'(\varphi)))(u)
\end{aligned}$$

Therefore $i'(\lambda\varphi) = \lambda i'(\varphi)$

Next we will show null $i' = U^0$

we have

$$\begin{aligned}
\text{null } i &= \{\varphi \in V' : i'(\varphi) = 0\} \\
&= \{\varphi \in V' : \varphi \circ i = 0\} \\
&= \{\varphi \in V' : (\varphi \circ i)(u) = 0 \text{ for all } u \in U\} \\
&= \{\varphi \in V' : \varphi(i(u)) = 0 \text{ for all } u \in U\} \\
&= \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\} \\
&= U^0
\end{aligned}$$

By the Fundamental Theorem of Linear Maps (3.22 of Axler) we have

$$\begin{aligned}
\dim V &= \dim V' && \text{by 3.95 of Axler} \\
&= \dim \operatorname{range} i' + \dim \operatorname{null} i' && \text{Fund. Thm of Linear Maps (3.22 of Axler)} \\
&= \dim \operatorname{range} i' + \dim U^0 \\
&= \dim U' + \dim U^0 \\
&= \dim U + \dim U^0
\end{aligned}$$

once we show range $i' = U$

Show range $i' = U'$

Suppose we have $\varphi \in U'$. By Exercise 3.A11 of Axler, we can extend to a linear functional ψ on V. And by definition of i' we have $i'(\psi) = \varphi$. So $\varphi \in$ range i', and so we have $U' \subset$ range i'. But 3.19 of Axler says that range i' is a subspace of U'.

Therefore, range $i' = U'$, as desired ▨

3.107 The null space of T'

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$

Then:

(a) null $T' = (\text{range } T)^0$

(b) dim null $T' =$ dim null T + dim W - dim V

Proof: (a) Suppose $\varphi \in$ null T'. Then we have $T'(\varphi) = 0$. So for all $v \in V$ we have

$$\begin{aligned}
0 &= 0(v) \\
&= (T'(\varphi))(v) \\
&= (\varphi \circ T)(v) \\
&= \varphi(Tv)
\end{aligned}$$

so $\varphi \in (\text{range } T)^0$, and so we have null $T' \subset (\text{range } T)^0$

Suppose $\varphi \in (\text{range } T)^0$. Then $\varphi(Tv) = 0$ for all $v \in V$.

So we have

$$\begin{aligned}
(T'(\varphi))(v) &= (\varphi \circ T)(v) \\
&= \varphi(Tv) \\
&= \varphi(0) \\
&= 0
\end{aligned}$$

and so $\varphi \in$ null T'. So $(\text{range } T)^0 \subset$ null T'

Therefore, $(\text{range } T)^0 =$ null T'

(b): we have

$$\begin{aligned}
\dim \operatorname{null} T' &= \dim (\operatorname{range} T)^0 && \text{by part (a)} \\
&= \dim W - \dim \operatorname{range} T && \text{by 3.106 of Axler} \\
&= \dim W - (\dim V - \dim \operatorname{null} T) && \text{by Fund Thm of Linear Maps (3.22 of Axler)} \\
&= \dim \operatorname{null} T + \dim W - \dim V
\end{aligned}$$

▨

3.108 T surjective is equivalent to T' injective

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$. Then T is surjective if and only if T' is injective.

Proof: $\iff$ = "iff and only if"

$$\begin{aligned} T \in \mathcal{L}(V,W) \text{ is surjective} &\iff \text{range } T = W \\ &\iff (\text{range } T)^0 = \{0\} \\ &\iff \text{null } T' = \{0\} \quad \text{by 3.107 of Axler} \\ &\iff T' \text{ is injective} \end{aligned}$$

Prove $(\text{range } T)^0 = \{0\}$

we have

$$\begin{aligned} \dim (\text{range } T)^0 &= \dim W - \dim \text{range } T \quad \text{by 3.106 of Axler} \\ &= \dim W - \dim W \\ &= 0 \\ &= \dim \{0\} \end{aligned}$$

$(\iff)$ if and only if $(\text{range } T)^0 = \{0\}$ by Exercise 2.C1 of Axler

3.109 The range of T'

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$. Then

(a) $\dim \text{range } T' = \dim \text{range } T$,

(b) $\text{range } T' = (\text{null } T)^0$

proof: (a) we have

$$\begin{aligned} \dim \text{range } T' &= \dim W' - \dim \text{null } T' \quad \text{by the Fund. Thm of Linear Maps (3.22 of Axler)} \\ &= \dim W - \dim \text{null } T' \quad \text{by 3.95 of Axler} \\ &= \dim W - \dim (\text{range } T)^0 \quad \text{by 3.107 (a) of Axler} \\ &= \dim \text{range } T \quad \text{by 3.106 of Axler} \end{aligned}$$

(b) Suppose $\varphi \in \text{range } T'$. Then there exists $\psi \in W'$ that satisfies $\varphi = T'(\psi)$. For all $v \in \text{null } T$, we have

$$\begin{aligned} \varphi(v) &= (T'(\psi))(v) \\ &= (\psi \circ T)(v) \\ &= \psi(Tv) \\ &= \psi(0) \quad \text{since } v \in \text{null } T \\ &= 0 \end{aligned}$$

So $\varphi \in (\text{null } T)^0$. Therefore, $\text{range } T' \subset (\text{null } T)^0$

Show $\dim \text{range } T' = \dim (\text{null } T)^0$

we have

$$\dim \text{range } T' = \dim \text{range } T \quad \text{by 3.109 (a) of Axler}$$

$= \dim V - \dim \operatorname{null} T$ by Fund Thm of Linear Maps (3.22 of Axler)

$= \dim (\operatorname{null} T)^0$ by 3.106 of Axler

Therefore, we have in fact range $T' = (\operatorname{null} T)^0$ ▨

3.110 T injective is equivalent to T' surjective

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then T is injective if and only if T' is surjective

Proof: $T \in \mathcal{L}(V, W)$ is injective $\Leftrightarrow$ $\operatorname{null} T = \{0\}$ by 3.16 of Axler

$\Leftrightarrow (\operatorname{null} T)^0 = V'$

$\Leftrightarrow$ range $T' = V'$ by 3.109 (b) of Axler

$\Leftrightarrow T'$ is surjective

Prove $(\operatorname{null} T)^0 = V'$

we have

$$\begin{aligned}\dim (\operatorname{null} T)^0 &= \dim V - \dim \operatorname{null} T \text{ by 3.106 of Axler}\\ &= \dim V - \dim \{0\}\\ &= \dim V - 0\\ &= \dim V\\ &= \dim V' \text{ by 3.95 of Axler}\end{aligned}$$

if and only if $(\operatorname{null} T)^0 = V'$, by Exercise 2.C.1 of Axler