3.F Duality

3.92 Definition

A *linear functional* on V is a linear map $\varphi: V \to \mathbb{F}$. In other words, $\varphi \in \mathcal{L}(V, \mathbb{F})$.

3.93 Example

- Define $\varphi: \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $\varphi(x,y,z) = 4x - 5y + 2z$. $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

  Then φ is a linear functional on $\mathbb{R}^3$.

- For some $c_1, \cdots, c_n \in \mathbb{F}$, then map $\varphi: \mathbb{F}^n \longrightarrow \mathbb{F}$ defined by $\varphi(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$,

  then φ is a linear functional on $\mathbb{F}^n$.

- Define $\varphi: P(\mathbb{R}) \longrightarrow \mathbb{R}$ by $\varphi(p) = \int_0^1 p(x)\,dx$.

  Then φ is a linear functional on $P(\mathbb{R})$.

3.94 Definition

The *dual space* of V, denoted V', is the vector space of all linear functionals on V. In other words,

$$V' = \mathcal{L}(V, \mathbb{F}).$$

3.95 $\dim V' = \dim V$

Suppose V is finite-dimensional. Then V' is also finite dimensional and

$$\dim V' = \dim V.$$

Proof:

$$\begin{aligned} \dim V' &= \dim \mathcal{L}(V, \mathbb{F}) \\ &= (\dim V)(\dim \mathbb{F}) \quad \text{by 3.61 of Axler} \\ &= (\dim V) \cdot 1 \\ &= \dim V. \end{aligned}$$

3.96 Definition

If $v_1, \ldots, v_n$ is a basis of V, then then dual basis of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j for any $j = 1, \ldots, n$ is a linear functional on V that satisfies

$$\varphi_j(v_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

3.97 Example

What is the dual basis of the standard basis $e_1, \ldots, e_n$ of $\mathbb{F}^n$?

$e_1 = (1, 0, \ldots, 0)$

$e_2 = (0, 1, 0, \ldots, 0)$

$e_3 = (0, 0, 1, 0, \ldots, 0)$

$\vdots$

$e_n = (0, \ldots, 0, 1)$

Group Exam

Solution: For all $j = 1, \ldots, n$, write $\varphi_j(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$

$c_1 = 1$ Then $1 = \varphi_1(v_1) = \varphi_1(e_1) = \varphi_1(1, 0, \ldots, 0) = c_1(1) + c_2(0) + \cdots + c_n(0) = c_1.$

$c_2 = 0$ $\quad 0 = \varphi_1(v_2) = \varphi_1(e_2) = \varphi_1(0, 1, 0, \ldots, 0) = c_1(0) + c_2(1) = c_3(0) + \cdots + c_n(0) = c_2$

$\vdots$ $\quad \vdots$

$c_n = 0$ $\quad 0 = \varphi_1(v_n) = \varphi_1(e_n) = \varphi_3(0, \ldots, 0, 1) = c_1(0) + \cdots + c_{n-1}(0) + c_n(1) = c_n$

So $\varphi_1(x_1, \ldots, x_n) = \overset{c_1}{(1)} x_1 + \overset{c_2}{(0)} x_2 + \cdots + \overset{c_n}{(0)} x_n$

$= x_1$

Similarly, $\varphi_1(x_1, \ldots, x_n) = x_1$

$\varphi_2(x_1, \ldots, x_n) = x_2$

$\varphi_3(x_1, \ldots, x_n) = x_3$

$\vdots$

$\varphi_n(x_1, \ldots, x_n) = x_n$

So $\varphi_1, \ldots, \varphi_n$ as defined above is the dual basis of $v_1, \ldots, v_n$.

Axler

Define φ_j to be the linear functional on $\mathbb{F}^n$ that selects the j^{th} coordinate of a vector in $\mathbb{F}^n$.

In other words, $\varphi_j(x_1, \dots, x_n) = x_j$ for all $(x_1, \dots, x_n) \in \mathbb{F}^n$

3.98 ~~Definition~~ Dual basis is a basis of the dual space.

Suppose V is finite-dimensional.

Then the dual basis of a basis of V is a basis of V'.

Proof: Let $v_1, \dots, v_n$ be a basis of V, and let $\varphi_1, \dots, \varphi_n$ be a dual basis of $v_1, \dots, v_n$.

We will show that $\varphi_1, \dots, \varphi_n$ is linearly independent.

Suppose $a_1, \dots, a_n \in \mathbb{F}$ satisfy $a_1\varphi_1 + \dots + a_n\varphi_n = 0$.

Then, for any $j = 1, \dots, n$, we have

$$
\begin{aligned}
(a_1\varphi_1 + \dots + a_n\varphi_n)(v_j) &= a_1\varphi_1(v_j) + \dots + a_n\varphi_n(v_j) \\
&= a_1\varphi_1(v_j) + \dots + a_j\varphi_j(v_j) + \dots + a_n\varphi_n(v_j) \\
&= a_1 \cdot 0 + \dots + a_j \cdot 1 + \dots + a_n \cdot 0 \\
&= a_j.
\end{aligned}
$$

So we have

$$
\begin{aligned}
a_j &= (a_1\varphi_1 + \dots + a_n\varphi_n)(v_j) \\
&= 0(v_j) \\
&= 0
\end{aligned}
$$

In other words, $a_1 = 0, \dots, a_n = 0$

So $\varphi_1, \dots, \varphi_n$ is linearly independent.

By 3.95 of Axler, $\dim V' = \dim V$.

By 2.39 of Axler, $\varphi_1, \dots, \varphi_n$ is a basis of V'.

3.99 Definition

If $T \in \mathcal{L}(V,W)$, then the <u>dual map</u> of T is the linear map $T' \in \mathcal{L}(W, V')$ defined by

$$T'(\varphi) = \varphi \circ T$$

input a linear functional

output a linear functional

<u>Proof</u>:

Show: $T' \in \mathcal{L}(W', V')$.

Let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in W'$ be arbitrary.

- <u>Additivity</u>:
$$\begin{aligned} T'(\varphi+\psi) &= (\varphi+\psi)\circ T \\ &= \varphi\circ T + \psi \circ T \\ &= T'(\varphi) + T'(\psi) \end{aligned}$$

- <u>Homogeneity</u>:
$$\begin{aligned} T'(\lambda\varphi) &= (\lambda\varphi)\circ T \\ &= \lambda(\varphi \circ T) \\ &= \lambda T'(\varphi). \end{aligned}$$

3.100 Example

Define $D : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$ by $Dp = p'$

- Define $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ by $\varphi(p) = p(3)$

  Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ that satisfies
$$\begin{aligned} (D'(\varphi))(p) &= (\varphi \circ D)(p) \\ &= \varphi(Dp) \\ &= \varphi(p') \\ &= p'(3). \end{aligned}$$

- Define $\varphi \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{F})$ by $\varphi(p) = \int_0^1 p(x)\,dx$

  Then $D'(\varphi)$ is the linear functional on $\mathcal{P}(\mathbb{R})$ given by

$$
\begin{aligned}
(D'(\varphi))(p) &= (\varphi \circ D)(p) \\
&= \varphi(Dp) \\
&= \varphi(p') \\
&= \int_0^1 p'(x)\,dx \\
&= p(1) - p(0).
\end{aligned}
$$

<u>3.101 Algebraic Properties of Dual Maps</u>

(a) $(S+T)' = S' + T'$ for all $S, T \in \mathcal{L}(V, W)$

(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$ and for all $T \in \mathcal{L}(V, W)$.

(c) $(ST)' = T'S'$ for all $T \in \mathcal{L}(U, V)$ and for all $S \in \mathcal{L}(V, W)$

<u>Proof</u>: (a): For all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$
\begin{aligned}
(S+T)'(\varphi) &= \varphi \circ (S+T) \\
&= \varphi \circ S + \varphi \circ T \\
&= S'(\varphi) + T'(\varphi).
\end{aligned}
$$

(b): For all $\varphi \in \mathcal{L}(V, \mathbb{F})$,

$$
\begin{aligned}
(\lambda T)'(\varphi) &= \varphi \circ (\lambda I) \\
&= \lambda \varphi \circ T \\
&= \lambda T'(\varphi)
\end{aligned}
$$

(c) For all $\varphi \in \mathcal{L}(V, \mathbb{F})$, we have

$$
\begin{aligned}
(ST)'(\varphi) &= \varphi \circ (ST) \\
&= (\varphi \circ S) \circ T \\
&= T'(\varphi \circ S) \\
&= T'(S'(\varphi)) \\
&= (T'S')(\varphi).
\end{aligned}
$$

So $(ST)'(\varphi) = T'S'$.

3.102 Definition

If U is a subspace of V, then the annihilator of U, denoted U^0, is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}$$

3.103 Example

Let U be the subspace of $P(\mathbb{R})$ consisting of all polynomials multiples of x^2, such as $x^2 p(x)$.

If $\varphi \in \mathcal{L}(P(\mathbb{R}), \mathbb{F})$ is defined by $\varphi(p) = p'(0)$, then $\varphi \in U^0$.

$x^2 p(x) \in U$, if $p \in P(\mathbb{R})$

And
$$\begin{aligned}\varphi(x^2 p(x)) &= (x^2 p(x))'\big|_{x=0} \\ &= (2x p(x) + x^2 p'(x))\big|_{x=0} \\ &= 2(0)p(0) + 0^2 p'(0) \\ &= 0\end{aligned}$$

3.104 Example

Let e_1, e_2, e_3, e_4, e_5 denote the standard basis of $\mathbb{R}^5$, and let $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ denote the dual basis of $(\mathbb{R}^5)'$.

Suppose
$$\begin{aligned}U &= \operatorname{span}(e_1, e_2) \\ &= \{(x_1, x_2, 0, 0, 0) \in \mathbb{R}^5 : x_1, x_2 \in \mathbb{R}\}.\end{aligned}$$

Show $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

Solution: Recall from Example 3.97 that $\varphi_j(x_1, x_2, x_3, x_4, x_5) = x_j$ for any $j = 1, 2, 3, 4, 5$.

Suppose we have $\varphi \in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$.

Then $\varphi = c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$

for some $c_3, c_4, c_5 \in \mathbb{R}$. For all $(x_1, x_2, 0, 0, 0) \in U$,

we have
$$
\begin{aligned}
\varphi(x_1, x_2, 0, 0, 0) &= (c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(x_1, x_2, 0, 0, 0) \\
&= c_3\varphi_3(x_1, x_2, 0, 0, 0) + c_4\varphi_4(x_1, x_2, 0, 0, 0) \\
&\quad + c_5\varphi_5(x_1, x_2, 0, 0, 0). \\
&= c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
&= 0
\end{aligned}
$$

So $\varphi \in U^0$, and so $\operatorname{span}(\varphi_3, \varphi_4, \varphi_5) \subset U^0$.

Suppose $\varphi \in U^0$. Since $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$ is the dual basis of $(\mathbb{R}^5)'$, we can write uniquely as

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5$$

for some $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$.

Since $e_1 \in U$, we have
$$
\begin{aligned}
0 &= \varphi(e_1) \\
&= (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_1) \\
&= c_1\varphi_1(e_1) + c_2\varphi_2(e_1) + c_3\varphi_3(e_1) + c_4\varphi_4(e_1) + c_5\varphi_5(e_1) \\
&= c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
&= c_1
\end{aligned}
$$

Since $e_2 \in U$, we have
$$
\begin{aligned}
0 &= \varphi(e_2) \\
&= (c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5)(e_2) \\
&= c_1\varphi_1(e_2) + c_2\varphi_2(e_2) + c_3\varphi_3(e_2) + c_4\varphi_4(e_2) + c_5\varphi_5(e_2) \\
&= c_1 \cdot 0 + c_2 \cdot 1 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
&= c_2
\end{aligned}
$$

Therefore,
$$
\begin{aligned}
\varphi &= c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \\
&= 0 \cdot \varphi_1 + 0 \cdot \varphi_2 + c_3\varphi_3 + c_4\varphi_4 + c_5\varphi_5 \\
&= c_3 \cdot \varphi_3 + c_4\varphi_4 + c_5 \cdot \varphi_5 \\
&\in \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)
\end{aligned}
$$

So $U^0 \subset \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

Therefore, $U^0 = \operatorname{span}(\varphi_3, \varphi_4, \varphi_5)$

3.105 The annihilator is a subspace

Suppose U is a subspace of V.

Then U^0 is a subspace of V'.

$$(V' = \mathcal{L}(V, \mathbb{F}))$$

Proof: • Additivity: Since $0(u) = 0$ for all $u \in U$,
we have $0 \in U^0$.

• Closed under addition: Suppose $\varphi, \psi \in U^0$. For all $u \in U$,
we have

$$(\varphi + \psi)(u) = \varphi(u) + \psi(u) = 0 + 0 = 0.$$

So $\varphi + \psi \in U^0$.

• Closed under scalar multiplication:

Suppose $\lambda \in \mathbb{F}$ and $\varphi \in U^0$. For all $u \in U$,
we have

$$(\lambda\varphi)(u) = \lambda\varphi(u) = \lambda \cdot 0 = 0.$$

So $\lambda\varphi \in U^0$.

3.111 Definition

The transpose of a matrix A, denoted A^t, is the matrix obtained from A by interchanging the rows and columns. More specifically, if

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \quad \text{then} \quad A^t = \begin{pmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{pmatrix}$$

3.112 Example

$$\text{If } A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix}, \quad \text{then } A^t = \begin{pmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{pmatrix}$$

3.113 The transpose of the product of matrices

Let A be an $m \times n$ matrix and β be an $n \times p$ matrix. Then

$$(AC)^t = C^t A^t$$

<u>Proof</u>: Suppose we have $k = 1, \ldots, p$ and $j = 1, \ldots, m$.

Then

$$
\begin{aligned}
((AC)^t)_{k,j} &= (AC)_{j,k} \\
&= \sum_{r=1}^{n} A_{j,r} C_{r,k} \\
&= \sum_{r=1}^{n} (A^t)_{r,j} (C^t)_{k,r} \\
&= \sum_{r=1}^{n} (C^t)_{k,r} (A^t)_{r,j} \\
&= (C^t A^t)_{k,j}
\end{aligned}
$$

Therefore, we have $(AC)^t = C^t A^t$

3.114 The matrix of T' is the transpose of the matrix of T

Suppose $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(T') = (\mathcal{M}(T))^t$.

<u>Proof</u>: Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$.

Suppose we have $j = 1, \ldots, m$ and $k = 1, \ldots, n$.

By Definition 3.32 of Axler, we have

$$T v_k = \sum_{r=1}^{m} A_{r,k} w_r$$

and

$$T'(\varphi_j) = \sum_{r=1}^{n} C_{r,j} \varphi_r .$$

So we have

$$
\begin{aligned}
(\varphi_j \circ T)(v_k) &= (T'(\varphi_j))(v_k) \\
&= \left(\sum_{r=1}^{n} C_{r,j} \varphi_r\right)(v_k) \\
&= (C_{1,j} \varphi_1 + \cdots + C_{n,j} \varphi_n)(v_k) \\
&= C_{1,j} \varphi_1(v_k) + \cdots + C_{n,j} \varphi_n(v_k) \\
&= C_{1,j} \cancelto{0}{\varphi_1(v_k)} + \cdots + C_{k,j} \cancelto{1}{\varphi_k(v_k)} + \cdots + C_{n,j} \cancelto{0}{\varphi_n(v_k)} \\
&= C_{1,j} \cdot 0 + \cdots + C_{k,j} \cdot 1 + \cdots + C_{n,j} \cdot 0 \\
&= C_{k,j}
\end{aligned}
$$

and $$\begin{aligned}(\varphi_j \circ T)(v_k) &= \varphi_j(Tv_k)\\ &= \varphi_j\left(\sum_{r=1}^{n} A_{r,k} w_r\right)\\ &= \varphi_j(A_{1,k}w_1 + \cdots + A_{n,k}w_n)\\ &= \varphi_j(A_{1,k}w_1) + \cdots + \varphi_j(A_{n,k}w_n)\\ &= A_{1,k}\varphi_j(w_1) + \cdots + A_{n,k}\varphi_j(w_n)\\ &= A_{1,k}\,\overset{0}{\cancel{\varphi_j(w_1)}} + \cdots + A_{j,k}\,\overset{1}{\cancel{\varphi_j(w_j)}} + \cdots + A_{n,k}\,\overset{0}{\cancel{\varphi_j(w_n)}}\\ &= A_{1,k}\cdot 0 + \cdots + A_{j,k}\cdot 1 + \cdots + A_{n,k}\cdot 0\\ &= A_{j,k}\end{aligned}$$

Therefore, we conclude $C_{k,j} = A_{j,k}$.

So $C = A^t$,

and so $$\begin{aligned}\mathcal{M}(T') &= C\\ &= A^t\\ &= (\mathcal{M}(T))^t.\end{aligned}$$

Back to Annihilator Stuff (3.106 – 3.110)

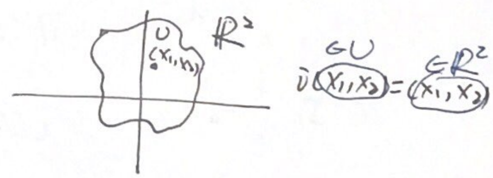
3.106 Dimension of the annihilator

Suppose V is a finite-dimensional vector space and U is a subspace of V. Then
$$\dim U + \dim U^0 = \dim V.$$

Proof: Let $i \in \mathcal{L}(U, V)$ be the inclusion map defined by $i(u) = u$ for all $u \in U$.

First, we will show that i' is linear.

Let $\lambda \in \mathbb{F}$ and $\varphi, \psi \in V'$.



• Additivity: For all $u \in U$, we have
$$\begin{aligned}(i'(\varphi+\psi))(u) &= ((\varphi+\psi)\circ i)(u)\\ &= (\varphi+\psi)(i(u))\\ &= (\varphi+\psi)(u)\\ &= \varphi(u) + \psi(u)\\ &= \varphi(i(u)) + \psi(i(u))\\ &= (\varphi \circ i)(u) + (\psi \circ i)(u)\\ &= (i'(\varphi))(u) + (i'(\psi))(u)\\ &= (i'(\varphi) + i'(\psi))(u)\end{aligned}$$

• Homogeneity: For all $u \in U$, we have

$$\begin{aligned}(i'(\lambda\varphi))(u) &= ((\lambda\varphi)\circ i)(u) \\ &= (\lambda\varphi)(i(u)) \\ &= (\lambda\varphi)(u) \\ &= \lambda\varphi(u) \\ &= \lambda\varphi(i(u)) \\ &= \lambda((\varphi\circ i)(u)) \\ &= (\lambda(i'(\varphi)))(u)\end{aligned}$$

Therefore, $i'(\lambda\varphi) = \lambda i'(\varphi)$.

Next, we will show null $i' = U^0$.

We have
$$\begin{aligned}\text{null } i' &= \{\varphi \in V' : i'(\varphi) = 0\} \\ &= \{\varphi \in V' : \varphi \circ i = 0\} \\ &= \{\varphi \in V' : \varphi(i(u)) = 0 \text{ for all } u \in U\}. \\ &= \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}. \\ &= U^0.\end{aligned}$$

By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned}\dim V &= \dim V' \quad \text{by 3.95 of Axler} \\ &= \dim \text{range } i' + \dim \text{null } i' \quad \text{Fund. Thm. of Linear Maps (3.22 of Axler)} \\ &= \dim \text{range } i' + \dim U^0 \\ &= \dim U' + \dim U^0 = \dim U + \dim U^0\end{aligned}$$

once we show range $i' = U$.

Show range $i' = U'$

Suppose we have $\varphi \in U'$. By Exercise 3.A.11 of Axler, we can extend to a linear functional ψ on V. And by definition of i', we have $i'(\psi) = \varphi$. So $\varphi \in$ range i', and so we have $U \subset$ rang i'. But 3.19 of Axler says that range i' is a subspace of U'.

Therefore, range $i' = U'$, as desired.

3.107 The null space of T'

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then:

(a) $\text{null } T' = (\text{range } T)^0$

(b) $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$.

<u>Proof</u>: (a) Suppose $\varphi \in \text{null } T'$. Then we have $T'(\varphi) = 0$.
So for all $v \in V$, we have

$$\begin{aligned} 0 &= 0(v) \\ &= (T'(\varphi))(v) \\ &= (\varphi \circ T)(v) \\ &= \varphi(Tv) \end{aligned}$$

So $\varphi \in (\text{range } T)^0$, and so we have $\text{null } T' \subset (\text{range } T)^0$.

Suppose $\varphi \in (\text{range } T)^0$. Then $\varphi(Tv) = 0$ for all $v \in V$.
So we have

$$\begin{aligned} (T'(\varphi))(v) &= (\varphi \circ T)(v) \\ &= \varphi(Tv) \\ &= \varphi(0) \\ &= 0 \end{aligned}$$

and so $\varphi \in \text{null } T'$. So $(\text{range } T)^0 \subset \text{null } T'$.
Therefore, $(\text{range } T)^0 = \text{null } T'$.

(b): We have

$$\begin{aligned} \dim \text{null } T' &= \dim (\text{range } T)^0 && \text{by part (a)} \\ &= \dim W - \dim \text{range } T && \text{by 3.106 of Axler} \\ &= \dim W - (\dim V - \dim \text{null } T) && \text{by Fund. Thm. of Linear Maps (3.22 of Axler)} \\ &= \dim \text{null } T + \dim W - \dim V. \end{aligned}$$

3.108 T surjective is equivalent to T' injective

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if T' is injective.

Proof: $\Longleftrightarrow$ = "if and only if"

$T \in \mathcal{L}(V,W)$ is surjective $\Longleftrightarrow$ range $T = W$

$\Longleftrightarrow$ $(\text{range } T)^0 = \{0\}$

$\Longleftrightarrow$ null $T' = \{0\}$ by 3.107(a) of Axler

$\Longleftrightarrow$ T' is injective

Proof $(\text{range } T)^0 = \{0\}$

we have

$$\begin{aligned}\dim (\text{range } T)^0 &= \dim W - \dim \text{range } T \quad \text{by 3.106 of Axler}\\ &= \dim W - \dim W\\ &= 0\\ &= \dim \{0\},\end{aligned}$$

if and only if $(\text{range } T)^0 = \{0\}$, by Exercise 2.C.1 of Axler.
($\Longleftrightarrow$)

3.109 The range of T'

Suppose V and W and finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$.

Then (a) dim range T' = dim range T,

(b) range $T' = (\text{null } T)^0$.

Proof: (a) we have

$$\begin{aligned}\dim \text{range } T' &= \dim W' - \dim \text{null } T' \quad \text{by the Fund. Thm. of Linear Maps (3.22 of Axler)}\\ &= \dim W - \dim \text{null } T' \quad \text{by 3.95 of Axler}\\ &= \dim W - \dim (\text{range } T)^0 \quad \text{by 3.107(a) of Axler}\\ &= \dim \text{range } T. \quad \text{by 3.106 of Axler.}\end{aligned}$$

(b) Suppose $\varphi \in$ range T'. Then there exists $\psi \in W'$ that satisfies $\varphi = T'(\psi)$. For all $v \in$ null T, we have

$$\begin{aligned}\varphi(v) = (T'(\psi))(v) &= \psi(0) \quad \text{since } v \in \text{null } T\\ = (\psi \circ T)(v) &= 0\\ = \psi(Tv)\end{aligned}$$

So $\varphi \in (\text{null } T)^\circ$. Therefore, range $T' \subset (\text{null } T)^\circ$.

Show dim range $T' = \dim (\text{null } T)^\circ$.

We have
$$\begin{aligned}\dim \text{range } T' &= \dim \text{range } T && \text{by 3.109 (a) of Axler}\\ &= \dim V - \dim \text{null } T && \text{by Fund. Thm. of Linear Maps (3.22 of Axler)}\\ &= \dim (\text{null } T)^\circ && \text{by 3.106 of Axler.}\end{aligned}$$

Therefore, we have in fact range $T' = (\text{null } T)^\circ$.

3.110 T injective is equivalent to T' surjective

Suppose V and W are finite-dimensional vector spaces and $T \in \mathcal{L}(V,W)$. Then T is injective if and only if T' is surjective.

Proof: $T \in \mathcal{L}(V,W)$ is injective
$$\begin{aligned}&\iff \text{null } T = \{0\} && \text{by 3.16 of Axler}\\ &\iff (\text{null } T)^\circ = V' \\ &\iff \text{range } T' = V' && \text{by 3.109 (b) of Axler}\\ &\iff T' \text{ is surjective}\end{aligned}$$

Prove $(\text{null } T)^\circ = V'$

We have
$$\begin{aligned}\dim (\text{null } T)^\circ &= \dim V - \dim \text{null } T && \text{by 3.106 of Axler}\\ &= \dim V - \dim \{0\}\\ &= \dim V - 0\\ &= \dim V\\ &= \dim V' && \text{by 3.95 of Axler}\end{aligned}$$

if and only if $(\text{null } T)^\circ = V'$, by Exercise 2.C.1 of Axler.