Extra Credit : Chapter 5.

Eigenvalue, Eigenvector, and Inariant Subspaces

5A. Invariant Subspaces

5.2 Definition

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called invariant under T if $u \in U$ implies $Tu \in U$.

5.3 example:
- $\{0\}$ if $u \in \{0\}$ then $u=0$

  So $Tu = T(0) = 0 \in \{0\}$

  So $\{0\}$ is invariant under T.
- V : $v \in V$, then $Tu \in V$

  So V is invariant under T
- null T : if $u \in \text{null}\, T$, then $Tu=0$

  So $T(Tu) = T(0) = 0$

  So $Tu \in \text{null}\, T$, and so null T is invariant under T.
- range T : if $u \in \text{range}\, T$, then $u = Tv$ for some $v \in V$.

  So $Tu = T(Tv)$

  Since $Tv \in V$, we get $Tu \in \text{range}\, T$

  So range T is invariant under T.

5.5 Eigenvalue

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an eigenvalue of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$

5.6 Equivalent conditions to be an eigenvalue

Suppose V is finite-dim, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ Then

(a) λ is an eigenvalue of T

(b) $T - \lambda I$ is not injective

(c) $T - \lambda I$ is not surjective

(d) $T - \lambda I$ is not invertible

Proof: Want: (a) is equivalent to (b): (a) $\to$ (b) and (b) $\to$ (a)

$Tv = \lambda v$ is equivalent to $(T - \lambda I)v = 0$

Since $v \neq 0$, we conclude that $T - \lambda I$ is not injective. Convesely, since $T - \lambda I$ is not injective $Tv = \lambda v$

So λ is an eigenvalue of T.

(b), (c), and (d) are equivalent to each other

By 3.69 of Axler, the following are equivalent.

$T - \lambda T$ is not invertible

$T - \lambda I$ is not surjective

and $T - \lambda I$ is not injective

5.7 eigenvector

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an eigenvector of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

5.8 Example Define $T \in \mathcal{L}(\mathbb{F}^2)$ by

$$T(w, z) = (-z, w)$$

Find the eigenvalues and eigenvectors of T.

Solution: let $\lambda \in \mathbb{F}$ satisify

$$T(w, z) = \lambda(w, z)$$

Since $T(w, z) = (-z, w)$ we obtain

$$(-z, w) = \lambda(w, z) = (\lambda w, \lambda z)$$

From $(-z, w) = (\lambda w, \lambda z)$ we got a system of equation

$$-z = \lambda w$$

$$w = \lambda z$$

Substitute $w = \lambda z$ into $-z = \lambda w$, we get

$$-z = \lambda w$$
$$= \lambda(\lambda z)$$
$$= \lambda^2 z$$

If $z=0$, then $w=0$, so $(w,z)=(0,0)$. If (w,z) is an eigenvector of T, then $z\neq 0$.

So $-z=\lambda^2 z$ implies $-1=\lambda^2$, on

$$\lambda = i,\ \lambda=-i$$

So $i, -i$ are our eigenvalues of T.

$\lambda = i$

$T(w,z) = i(w,z)$

implies $(-z, w) = (iw, iz)$

So $-z = iw$

$w = iz$

Thus, $(w,z) = (w, -iw)$

is an eigenvector of T corresponding to $\lambda = i$ for all $w\in\mathbb{C}$

$\lambda = i$

$T(w,z) = -i(w,z)$ implies $(-z,w) = (-iw, -iz)$

So, $-z=-iw$

$w=-iz$

Therefore, $(w,z) = (w, iw)$ is an eigenvector of T corresponding to $\lambda=-i$ for all $w\in\mathbb{C}$

Alternate method: Find $M(T) = M\left(T, \underset{\text{of } \mathbb{C}^2}{\text{standard basis}}, \underset{\text{of } \mathbb{C}^2}{\text{standard basis}}\right)$

Find $a,b,c,d\in\mathbb{F}$ that satisfy: $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} w \\ z\end{pmatrix} = \begin{pmatrix} -z \\ w\end{pmatrix}$

$$\begin{pmatrix} aw+bz \\ cw+dz\end{pmatrix} = \begin{pmatrix} -z \\ w\end{pmatrix}$$

$a=0 \quad b=-1$

$c=1 \quad d=0$

So $\mathcal{M}(T) = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$

$$\mathcal{M}(T-\lambda I) = M(T) - \lambda M(T)$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda\end{pmatrix}$$

$$\det(\mathcal{M}(T-\lambda I))=0$$

$$\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}=0$$

$$\lambda^2+1=0$$

$$\lambda=i,\ \lambda=-i$$

$$\mathcal{M}(T-iI)(w,z)=(0,0)$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-iw-z=0$$

$$w-iz=0$$

$$-z=iw$$

$$w=iz$$

$$(w,z)=(w,-iw)$$

$$\mathcal{M}(T-(-i)I)(w,z)=(0,0)$$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

5.10 Linearly independent eigenvectors

Let $T\in\mathcal{L}(V)$ Suppose $\lambda_1,\dots,\lambda_m$ are distinct eigenvalues of T and $v_1,\dots,v_m$ are corresponding eigenvectors. Then $v_1,\dots,v_m$ is linearly independent.

<u>Proof:</u> Suppose $v_1,\dots v_m$ is linearly Dependent.

Let k be the smallest positive integer such that.

$$v_k\in\operatorname{span}(v_1,\dots v_{k-1})$$

So there exist $a_1,\dots a_{k-1}\in\mathbb{F}$ such that

$$v_k=a_1v_1+\cdots+a_{k-1}v_{k-1}$$

So we have
$$\begin{aligned}Tv_k&=T(a_1v_1+\cdots+a_{k-1}v_{k-1})\\&=T(a_1v_1)+\cdots+T(a_{k-1}v_{k-1})\\&=a_1T(v_1)+\cdots+a_{k-1}T(v_{k-1})\\&=a_1(\lambda_1v_1)+\cdots+a_{k-1}(\lambda_{k-1}v_{k-1})\\&=a_1\lambda_1v_1+\cdots+a_{k-1}\lambda_{k-1}v_{k-1}\end{aligned}$$

Multiply by λ_k both side of $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$ to get

$$Tv_k = \lambda_k v_k = a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1}$$

So $$0 = Tv_k - Tv_k$$

$$= (a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}) - (a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1})$$

$$= a_1 (\lambda_1 - \lambda_k) v_1 + \cdots + a_{k-1} (\lambda_{k-1} - \lambda_k) v_{k-1}$$

Since k is the smallest positive integer for which $v_k \in \operatorname{span}(v_1 \ldots v_{k-1})$

it follows that $v_1, \ldots v_{k-1}$ is linearly independent

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