Extra Credit Lecture : Chapter 5

Eigenvalues, Eigenvectors, and Invariant subspaces

5.A Invariant subspaces

5.2 $\boxed{\text{Def}}$

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called invariant under T if $u \in U$ implies $Tu \in U$,

5.3 Example

- $\{0\}$: If $u \in \{0\}$, then $u = 0$. So we have
  $Tu = T(0) = 0 \in \{0\}$
  So $\{0\}$ is invariant under T
- V: If $u \in V$, then $Tu \in V$.
  So V is invariant under T.
- null T: if $u \in \text{null } T$, then $Tu = 0$. So we have
  $T(Tu) = T(0) = 0$
  So $Tu \in \text{null } T$, and so null T is invariant under T.
- range T: If $u \in \text{range } T$, then $u = Tv$ for some $v \in V$.
  So we have $Tu = T(Tv)$.
  Since $Tv \in V$, we got $Tu \in \text{range } T$,
  So range T is invariant under T.

Eigenvalues & eigenvectors

5.5 $\boxed{\text{def}}$

Suppose $T \in \mathcal{L}(V)$, A scalar $\lambda \in \mathbb{F}$ is called eigenvalue of T if $\exists\ v \in V$ such that $v \neq 0$ & $Tv = \lambda v$

Such a vector $v \in V$ is called an eigenvector of T corresponding to the eigenvalue λ.

5.6 Equivalent conditions to be an eigenvalue

Suppose V is finite-dim, $T \in \mathcal{L}(V)$, & $\lambda \in \mathbb{F}$. Then the following are equivalent:

ⓐ : λ is an eigenvalue of T
ⓑ : $T-\lambda I$ is not injective
ⓒ : $T-\lambda I$ is not surjective
ⓓ : $T-\lambda I$ is not invertible

Proof:
ⓐ $\Leftrightarrow$ ⓑ (ⓐ implies ⓑ and ⓑ implies ⓐ)
The equation $Tv=\lambda v$ is equivalent to $(T-\lambda I)v=0$.
ⓑ, ⓒ, ⓓ are equivalent to each other.
By 3.69, the following are equivalent:
$T-\lambda I$ is not invertible,
$T-\lambda I$ is not injective,
$T-\lambda I$ is not surjective.

5.7 (Def)
Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T.
A vector $v \in V$ is called an eigenvector of T corresponding to λ if $v \neq 0$ and $Tv=\lambda v$.

5.8 Example
Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w,z)=(-z,w)$
Find the eigenvalues & eigenvectors of T.
Solution: let $\lambda \in \mathbb{F}$ satisfy $T(w,z)=\lambda(w,z)$
Since $T(w,z)=(-z,w)$, we obtain $(-z,w)=\lambda(w,z)$
$=(\lambda w, \lambda z)$
From $(-z,w)=(\lambda w,\lambda z)$, we got a system of equations
$-z=\lambda w$, $w=\lambda z$.
Substitute $w=\lambda z$ into $-z=\lambda w$, we got $-z=\lambda w$
$=\lambda(\lambda z)$
$=\lambda^2 z$

If $z=0$, then $w=0$. If (w,z) is an eigenvector of T, then $z\neq 0$. So $-z=\lambda^2 z$ implies $-1=\lambda^2$, or $\lambda=i, \lambda=-i$
So $i, -i$ are the eigenvalues of T.

- $\lambda = i$ $T(w,z) = i(w,z)$ implies $(-z,w) = (iw, iz)$

So $-z = iw$, $w = iz$

Therefore $(w,z) = (w,-iw)$

is an eigenvector of T $\forall$ $w \in \mathbb{C}$.

- $\lambda = -i$ $T(w,z) = -i(w,z)$ implies $(-z,w) = (-iw,-iz)$

So $-z = -iw$ $w = -iz$

So $(w,z) = (w, iw)$ is an eigenvector of T corresponding

to $\lambda = -i$ $\forall w \in \mathbb{C}$.

Alternate method.

Find, $M(T) = M(T$, standard basis of $\mathbb{C}^2$, standard basis of $\mathbb{C}^2)$.

Find $a, b, c, d \in \mathbb{F}$ satisfy

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} w \\ z \end{vmatrix} = \begin{vmatrix} -z \\ w \end{vmatrix}$$

$$\begin{vmatrix} aw + bz \\ cw + dz \end{vmatrix} = \begin{vmatrix} -z \\ w \end{vmatrix}$$

$a = 0$ $b = -1$

$c = 1$ $d = 0$

$$\begin{vmatrix} 0w + (-1)z \\ 1w + 0z \end{vmatrix} = \begin{vmatrix} -z \\ w \end{vmatrix}$$

$$M(T - \lambda I) = M(T) - \lambda M(I)$$
$$= \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix}$$

$\det(M(T - \lambda I)) = 0$

$$\det \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \qquad \lambda^2 + 1 = 0$$

$\lambda = i, \lambda = -i$

$M(T - iI)(w,z) = (0,0)$

$$\begin{vmatrix} -i & -1 \\ 1 & -i \end{vmatrix} \begin{vmatrix} w \\ z \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$-iw - z = 0$ $\quad w + iz = 0$

$-z = iw$

$w = -iz$ $\qquad (w,z) = (w, -iw)$

$M(T-(-i)I)(w,z)=(0,v)$

$$\begin{vmatrix} i & -1 \\ 1 & i \end{vmatrix}\begin{vmatrix} w \\ z \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$iw - z = 0$

$w + iz = 0$

$z = iw$

$w = iz$

$(w,z) = (w, iw)$

5.10 linearly independent eigenvectors

let $T \in \mathcal{L}(V)$ Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigen values of T and $v_1, \dots, v_m$ are the corresponding eigenvectors. Then $v_1, \dots, v_m$ is ℓI.

Proof: Suppose by contradiction, $v_1, \dots v_m$ is ℓD.

By ℓD lemma (2.21) $\exists$ the smallest positive int $k \in \{1, \dots, m\}$ such that $v_k \in span(v_1, \dots, v_{k-1})$

So $\exists$ $a_1, \dots, a_{k-1} \in \mathbb{F}$ such that $v_k = a_1 v_1 + \dots + a_{k-1} v_{k-1}$

So we have $Tv_k = T(a_1 v_1 + \dots + a_{k-1} v_{k-1})$

$= a_1 T v_1 + \dots + a_{k-1} T v_{k-1} = a_1 (\lambda v_1) + \dots + a_{k-1} (\lambda v_{k-1})$

$= a_1 \lambda v_1 + \dots + a_{k-1} \lambda v_{k-1}$

Multiply by λ_k both sides of $v_k = a_1 v_1 + \dots a_{k-1} v_{k-1}$ to write $Tv_k = \lambda_k v_k = a_1 \lambda_k v_1 + \dots a_{k-1} \lambda_k v_{k-1}$

So we get $0 = Tv_k - Tv_k$

$= (a_1 \lambda_1 v_1 + \dots + a_{k-1} \lambda_{k-1} v_{k-1}) -$

$(a_1 \lambda_k v_1 + \dots + a_{k-1} \lambda_{k-1} v_{k-1})$

$= a_1 (\lambda_1 - \lambda_k) v_1 + \dots + a_{k-1} (\lambda_{k-1} - \lambda_k) v_{k-1}$

Since k is the smallest positive int. for which $v_k \in span(v_1, \dots, v_{k-1})$ it follows that $v_1, \dots, v_{k-1}$ is ℓI.

So $a_1 (\lambda_1 - \lambda_k) = 0, \dots, a_{k-1} (\lambda_{k-1} \; \lambda_k) = 0$

But eigen values are distinct, so $a_1 = 0, \dots, a_{k-1} = 0$

So each $v_k = 0$, which contradicts our hypothesis that v_k is a eigen vector.
So $v_1, \ldots, v_m$ is l.I. $\blacksquare$