# Extra Credit Lecture

Chapter 5 Eigenvalues, Eigenvectors & Inariant Subspaces

5.A. Invarient Subspaces

5.2 Definition

Suppose $T \in \mathcal{L}(V)$. A subspace $U$ of $V$ is called invarient under $T$ if $u \in U$ implies $Tu \in U$.

5.3 Example

- $\{0\}$: If $u \in \{0\}$, then $u = 0$. So we have $Tu = T(0)$
$= 0$
$\in \{0\}$

So $\{0\}$ is invarient under $T$.

- $V$: If $u \in V$, then $Tu \in V$, so $V$ is invarient under $T$.
- null $T$: If $u \in \text{null}\,T$, then $Tu = 0$, so we have $T(Tu) = T(0)$
$= 0$

So $Tu \in \text{null}\,T$, and so null $T$ is invarient under $T$

- range $T$: If $u \in \text{range}\,T$, then $u = Tv$ for some $v \in V$.

So we have $Tu = T(Tv)$. Since $Tv \in V$, we get $Tu \in \text{range}\,T$. So range $T$ is invarient under $T$.

Eigenvalues & Eigenvectors

5.5 Definition

Suppose $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is called an eigenvalve of $T$ if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Such a vector $v \in V$ is called an eigenvector of $T$ corresponding to the eigenvalve $\lambda$.

5.6 Equivalent conditions to be an eigenvalue

Suppose V is finite dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

a) $\lambda$ is an eigenvalue of T

b) $T - \lambda I$ is not injective

c) $T - \lambda I$ is not surjective

d) $T - \lambda I$ is not invertible

**Proof:**

(a) is equivalent to (b) ((a) implies (b), & (b) implies (a))

The equation $Tv = \lambda v$ is equivalent to the equation $(T - \lambda I)v = 0$

$Tv = \lambda v$ iff $Tv - \lambda v = 0$ iff $Tv - \lambda I v = 0$ iff $(T - \lambda I)(v) = 0$, $v \neq 0$ so $(T - \lambda I)$ not injective

(b) (c) (d) are equivalent to each other

By 3.69 of Axler, the following are equivalent:

1) $T - \lambda I$ is not invertible

2) $T - \lambda I$ is not injective

3) $T - \lambda I$ is not surjective

5.7 Definition

Suppose $T \in \mathcal{L}(V)$ & $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an eigenvector of $T_v$ corresponding to $\lambda$ if $v \neq 0$ & $Tv = \lambda v$

5.8 Example

Define ~~[illegible]~~ $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w, z) = (-z, w)$

Find the eigen values & eigenvectors of T

Solution:

Let $\lambda \in \mathbb{F}$ satisfy $T(w,z) = \lambda(w,z)$

Since $T(w,z) = (-z,w)$, we obtain $(-z,w) = \lambda(w,z)$

$$= (\lambda w, \lambda z)$$

From $(-z,w) = (\lambda w, \lambda z)$ we get a system of equations

$$-z = \lambda w \quad \text{and} \quad w = \lambda z$$

Substitute $w = \lambda z$ into $-z = \lambda w$, we get

$$-z = \lambda w$$
$$= \lambda(\lambda z)$$
$$= \lambda^2 z$$

If $z = 0$, then $w = 0$, & $(w,z) = (0,0)$. If $(w,z)$ is an eigenvector of $T$, then $z \neq 0$ so $-z = \lambda^2 z$ implies $-1 = \lambda^2$ or $\lambda = i$, $\lambda = -i$

So $i, -i$ are our eigenvalues of $T$

$\lambda = i$

$T(w,z) = i(w,z)$

$(-z,w) = (iw, iz)$

So $-z = iw$

$w = iz$

Therefore,

$(w,z) = (w, -iw)$

is an eigenvector of $T$ for all $w \in \mathbb{C}$, corresponding to $\lambda = i$

$\lambda = -i$

$T(w,z) = -i(w,z)$

$(-z,w) = (-iw, -iz)$

So $-z = -iw$

$w = -iz$

Therefore,

$(w,z) = (w, iw)$

is an eigenvector of $T$ corresponding to $\lambda = -i$ for all $w \in \mathbb{C}$

Alternate method: Find $\mathcal{M}(T) = \mathcal{M}(T, \text{standard basis of } \mathbb{C}^2, \text{standard basis of } \mathbb{C}^2)$

Find $a, b, c, d \in \mathbb{F}$ that satisfy $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$

$\begin{pmatrix} aw + bz \\ cw + dz \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$

$a = 0, b = -1, c = 1, d = 0$ $\quad \begin{pmatrix} 0w + (-1)z \\ 1w + 0z \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$

So $M(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow M(T) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$M(T - \lambda I) = M(T) - \lambda M(I)$

$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -\lambda & -1 \\ -1 & -\lambda \end{pmatrix}$

$\det(M(T - \lambda I)) = 0$

$\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$

$\lambda^2 + 1 = 0$

$\lambda = i \quad \lambda = -i$

$M(T - iI)(w, z) = (0, 0)$ $\qquad$ $M(T - (-i)I)(w, z) = (0, 0)$

$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-iw - z = 0$ $\qquad$ $iw - z = 0$

$w + iz = 0$ $\qquad$ $w + iz = 0$

$-z = iw \quad w = iz$ $\qquad$ $z = iw \quad w = -iz$

$(w, z) = (w, -iw)$ $\qquad$ $(w, z) = (w, iw)$

5.10 Linearly independant eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvecters of $T$ & $v_1, \ldots, v_m$ are the corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly dependant.

Proof:

Suppose by contradiction that $v_1, \dots, v_m$ is linearly dependant.
By the Linear Dependence Lemma (2.21 of Axler), there exists the smallest positive integer $k \in \{1, \dots, m\}$ such that $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$
So there exist $a_1, \dots, a_{k-1} \in \mathbb{F}$ such that $v_k = a_1 v_1 + \dots + a_{k-1} v_{k-1}$
So we have

$$Tv_k = T(a_1 v_1 + \dots + a_{k-1} v_{k-1})$$
$$= a_1 T v_1 + \dots + a_{k-1} T v_{k-1}$$
$$(\star) \quad = a_1(\lambda v_1) + \dots + a_{k-1}(\lambda v_{k-1})$$

Multiply by $\lambda_k$ both sides of $v_k = a_1 v_1 + \dots + a_{k-1} v_{k-1}$ to write

$$\boxed{Tv_k = \lambda_k v_k} = a_1 \lambda_k v_1 + \dots + a_{k-1} \lambda_k v_{k-1} = \star$$

because $v_k$ is an eigenvector of T

$$\star - \star$$

So we get

$$0 = Tv_k - Tv_k$$
$$= (a_1 \lambda_1 v_1 + \dots + a_{k-1} \lambda_{k-1} v_{k-1}) - (a_1 \lambda_k v_1 + \dots + a_{k-1} \lambda_k v_{k-1})$$
$$= a_1(\lambda_1 - \lambda_k) v_1 + \dots + a_{k-1}(\lambda_{k-1} - \lambda_k) v_{k-1}$$

Since k is the smallest positive integer for which $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$ it follows that $v_1, \dots, v_{k-1}$ is linearly independant. So we get

$$a_1(\lambda_1 - \lambda_k) = 0, \dots, a_{k-1}(\lambda_{k-1} - \lambda_k) = 0$$

But $\lambda_1, \dots, \lambda_m$ are distinct, So $a_1 = 0, \dots,$ ~~and~~ $a_{k-1} = 0$
So each $v_k$ is zero, which contradicts our hypothesis that $v_k$ is an eigenvector. So $v_1, \dots, v_m$ is linearly independant. □