Extra lecture : Chapter 5

Eigen values. Eigen vectors, and Invariant Subspaces

5A. Invarian subspaces.

5.2 Definition.

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called invariant under T if $u \in U$ implies $Tu \in U$.

5.3 Example

- $\{0\}$: if $u \in \{0\}$, then $u = 0$. So we have

$$T(u) = T(0) = 0 \in \{0\}$$

So $\{0\}$ is invariant under T.

- V: If $u \in V$, Then $Tu \in V$

So V is invariant under T.

- null T: If $u \in$ null T, then $Tu = 0$. So we have:

$$T(Tu) = T(0) = 0$$

$\therefore$ $Tu \in$ null T, and so null T is invariant under T.

- range T: If $u \in$ range T, then $u = Tv$ for some $v \in V$.

So we have:

$$Tu = T(Tv)$$

Since $Tv \in V$, we get $Tu \in$ range T.

$\therefore$ T is invariant under T.

Eigenvalues and eigenvectors

5.5 Definition

Suppose $T \in \mathcal{L}(V)$, A scalar $\lambda \in \mathbb{F}$ is called an eigenvalue of T if there exists $v \in V$ s.t. $v \neq 0$ and $Tv = \lambda v$.

Such a vector $v \in V$ is called an eigenvector of T corresponding to the eigen value of λ

5.6 Equivalent conditions to be an eigenvalue.

Suppose V is finite-dimensional. $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$.

Then the following are equivalent:

a). λ is an eigen value of T.

b). $T - \lambda I$ is not injective.

c). $T - \lambda I$ is not surjective

d). $T - \lambda I$ is not invertible.

Proof: (a) is equivalent to (b): a implies b, b implies a.

The equation $Tv = \lambda v$ is equivalent to the equation.

$$(T - \lambda I)v = 0$$

$$\begin{aligned} Tv = \lambda v \quad & \text{iff} \quad Tv - \lambda v = 0 \\ & \text{iff} \quad Tv - \lambda I v = 0 \\ & \text{iff} \quad (T - \lambda I)v = 0 \end{aligned}$$

(b). (c). (d) are equivalent to each other

By 3.69, the following are equivalent:

$T - \lambda I$ is not invertible.

$T - \lambda I$ is not injective,

$T - \lambda I$ is not surjective.

b implies a.

Conversely, $\because T - \lambda$ is not injective, $Tv = \lambda v$.

So λ is an eigenvalue of T.

5.7 Definition

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an eigenvector of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

5.8 Example

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by

$$T(w, z) = (-z, w)$$

Find the eigenvalues and eigenvectors of T.

Solution: Let $\lambda \in \mathbb{F}$ satisfy

$$T(w, z) = \lambda(w, z)$$

Since $T(w, z) = (-z, w)$, we obtain

$$\begin{aligned}(-z, w) &= \lambda(w, z) \\ &= (\lambda w, \lambda z)\end{aligned}$$

From $(-z, w) = (\lambda w, \lambda z)$, we get a system of equations.

$$-z = \lambda w$$

$$w = \lambda z$$

Substitute $w = \lambda z$ into $-z = \lambda w$, we get

$$\begin{aligned}-z &= \lambda w \\ &= \lambda(\lambda z)\end{aligned}$$

$=\lambda^2 z$

If $z=0$, then $w=0$, so $(w,z)=(0,0)$. If (w,z) is an eigenvector of T, then $z\neq 0$

So $-z=\lambda^2 z$ imples $-1=\lambda^2$ or

$$\lambda=i,\ \lambda=-i$$

So $i,-i$ are our eigenvalues of T.

$\lambda=i$

$T(w,z)=i(w,z)$ implies

$(-z,w)=(iw,iz)$

So

$-z=iw$

$w=iz$

$\therefore (w,z)=(w,-iw)$ is an eigenvector of T for all $w\in\mathbb{C}$.

$\lambda=-i$

$T(w,z)=-i(w,z)$

implies $(-z,w)=(-iw,-iz)$

So $-z=-iw$, $w=-iz$

$\therefore (w,z)=(w,iw)$ is an eigenvector of T corresponding to $\lambda=-i$, for all $w\in\mathbb{C}$.

Alternate method: $\mathbb{F}(n)$

$M(T)=M(T$, standard basis of $\mathbb{C}^2$, standard basis of $\mathbb{C}^2)$

Find $a,b,c,d\in\mathbb{F}$ that satisfy:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix}=\begin{bmatrix} -z \\ w \end{bmatrix}$$

$$\begin{bmatrix} aw+bz \\ cw+dz \end{bmatrix} = \begin{bmatrix} -z \\ w \end{bmatrix}$$

$a=0 \quad b=-1$

$c=1 \quad d=0$

$$M(T-\lambda T) = M(T) - \lambda M(T)$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix}$$

$$\det(M(T-\lambda T)) = 0$$

$$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = i \text{ or } -i.$$

$$M(T-iI)(w,z) = (0,0)$$

$$\begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$iw - z = 0$$

$$w + iz = 0$$

$$-z = iw$$

$$w = iz$$

$$(w,z) = (w, -iw)$$

$$M(T-(-i)I)(w,z) = (0,0)$$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \; iw - z = 0$$

$$\bar{w} + iz = 0$$
$$z = iw$$
$$w = -iz$$
$$(w, z) = (w, iw)$$

5.13 Linear independent eigenvectors.

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T and $v_1, \dots, v_m$ are the corresponding eigenvectors. Then $v_1, \dots, v_m$ is linearly independent.

Proof:

Suppose by contradiction that $v_1, \dots, v_m$ is linearly dependent. By the linear dependence Lemma (2.21). there exists the smallest positive integer $k \in \{1, \dots, m\}$ s.t.

$$v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$$

So there exist $a_1, \dots, a_{k-1} \in \mathbb{F}$ s.t.

$$v_k = a_1 v_1 + \dots + a_{k-1} v_{k-1}$$

So we have

$$\begin{aligned} Tv_k &= T(a_1 v_1 + \dots + a_{k-1} v_{k-1}) \\ &= a_1 Tv_1 + \dots + a_{k-1} Tv_{k-1} \\ &= a_1 (\lambda v_1) + \dots + a_{k-1} (\lambda v_{k-1}) \\ &= a_1 \lambda v_1 + \dots + a_{k-1} \lambda v_{k-1} \end{aligned}$$

Multiply by λ_k both sides of $v_k = a_1 v_1 + \dots + a_{k-1} v_{k-1}$ to write

$$Tv_k = \lambda_k v_k = a_1 \lambda_k v_1 + \dots + a_{k-1} \lambda_k v_{k-1}$$

So we get

$$0 = Tv_k - Tv_k$$

$= (a_1\lambda_1 v_1 + \dots + a_{k-1}\lambda_{k-1} v_{k-1}) - (a_1\lambda_k v_1 + \dots + a_{k-1}\lambda_k v_{k-1})$

$= a_1(\lambda_1 - \lambda_k)v_1 + \dots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1}$

Since k is the smallest positive integer for which

$v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$

It follows that $v_1, \dots, v_{k-1}$ is linearly independent.

$a_1(\lambda_1 - \lambda_k) = 0, \dots, a_{k-1}(\lambda_{k-1} - \lambda_k) = 0$

But $\lambda_1, \dots, \lambda_m$ are distinct so $a_1 = 0, \dots, a_k = 0$

So each v_k is zero, which contradicts our hypothesis that v_k is an eigenvector.

So $v_1, \dots, v_m$ is linearly independent.