Extra Credit Lecture:

Chapter Five

Eigenvalues, Eigenvectors, and Invariant Subspaces

5.A Invariant Subspaces

5.2 Definition:

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called <u>invariant</u> under T if $u \in U$ implies $Tu \in U$.

5.3 Example

- $\{0\}$: If $u \in \{0\}$, then $u = 0$. So we have

$$\begin{aligned} Tu &= T(0) \\ &= 0 \\ &\in \{0\}. \end{aligned}$$

So $\{0\}$ is invariant under T.

- V: If $u \in V$, then $Tu \in V$.

  So V is invariant under T.

- null T: If $u \in \text{null } T$, then $Tu = 0$. So we have

$$\begin{aligned} T(Tu) &= T(0) \\ &= 0 \end{aligned}$$

So $Tu \in \text{null } T$ and so null T is invariant under T.

- range T: If $u \in \text{range } T$, then $u = Tv$ for some $v \in V$.

  So we have

$$Tu = T(Tv).$$

Since $Tv \in V$, we get $Tu \in \text{range } T$.

So range T is invariant under T.

Eigenvalues and eigenvectors

5.5 Definition:

Suppose $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is called an eigenvalue of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Such a vector $v \in V$ is called an eigenvector of T corresponding to the eigenvalue λ.

5.6 Equivalent conditions to be an eigenvalue

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

(a): λ is an eigenvalue of T
(b): $T - \lambda I$ is not injective
(c): $T - \lambda I$ is not surjective
(d): $T - \lambda I$ is not invertible

Proof: (a) is equivalent to (b): ((a) implies (b), and (b) implies (a))

(a) implies (b):

The equation $Tv = \lambda v$ is equivalent to the equation $T - \lambda I v = 0$.

$Tv = \lambda v$

iff $Tv - \lambda v = 0$

iff $Tv - \lambda I v = 0$

iff $(T - \lambda I)(v) = 0$.

(b), (c), (d) are equivalent to each other

Since $v \neq 0$, we conclude that $T - \lambda I$ is not injective. (b) Conversely, (b) implies (a) since $T - \lambda I$ is not injective, $Tv = \lambda v$, so λ is an eigenvalue of T. (a)

By 3.62 of Axler, the following are equivalent:

$T - \lambda I$ is not invertible,

$T - \lambda I$ is not injective,

and $T - \lambda I$ is not surjective.

5.7 Definition:

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an eigenvector of TV corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

5.8 Example | Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(w, z) = (-z, w)$.
Find the eigenvalues and eigenvectors of T.

Solution: Let $\lambda \in \mathbb{F}$ satisfy $T(w, z) = \lambda(w, z)$.
Since $T(w, z) = (-z, w)$, we obtain

$$(-z, w) = \lambda(w, z)$$
$$= (\lambda w, \lambda z).$$

From $(-z, w) = (\lambda w, \lambda z)$, we get a system of equations

$$-z = \lambda w$$
$$w = \lambda z$$

Substitute $w = \lambda z$ into $-z = \lambda w$, we get

$$-z = \lambda w$$
$$= \lambda(\lambda z)$$
$$= \lambda^2 z.$$

If $z = 0$, then $w = 0$. If (w, z) is an eigenvector of
so $(w, z) = (0, 0)$. T, then $z \neq 0$.

So $-z=\lambda^2 z$ implies $-1=\lambda^2$, or
$\lambda=i, \lambda=-i$.
So $i, -i$ are our eigenvalues of T.

$\underline{\lambda=i}$

$T(w,z)=i(w,z)$
implies
$(-z,w)=(iw,iz)$
So
$-z=iw$
$w=iz$.
Therefore
$(w,z)=(w,-iw)$
is an eigenvector of T (corresponding to $\lambda=i$)
for all $w\in\mathbb{C}$.

$\underline{\lambda=-i}$

$T(w,z)=-i(w,z)$
implies
$(-z,w)=(-iw,-iz)$
So
$-z=-iw$
$w=-iz$
Therefore
$(w,z)=(w,iw)$
is an eigenvector of T corresponding to $\lambda=-i$ for all $w\in\mathbb{C}$.

Alternative method: Find
$M(T)=M(T$, standard bases of $\mathbb{C}^2$, stand. basis of $\mathbb{C}^2)$
Find $a,b,c,d\in\mathbb{F}$ that satisfy
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix}=\begin{pmatrix} -z \\ w \end{pmatrix}$$
$$\begin{pmatrix} aw+bz \\ cw+dz \end{pmatrix}=\begin{pmatrix} -z \\ w \end{pmatrix}$$
$a=0$ $\quad b=-1$
$c=1$ $\quad d=0$

So $M(T)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$M(T)=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0w+(-1)z \\ 1w+0z \end{pmatrix}=\begin{pmatrix} -z \\ w \end{pmatrix}$$

$$M(T-\lambda I) = M(T) - \lambda M(I)$$
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$

$$\det(M(T-\lambda I)) = 0$$
$$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda = i, \ \lambda = -i$$

$$M(T - iI) \cancel{Ma} (w, z) = (0,0)$$
$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$-iw - z = 0$$
$$w \ast iz = 0$$

$$-z = iw$$
$$w = +iz$$

$$(w, z) = (w, -iw)$$

$$M(T - (-i)I)(w, z) = (0,0)$$
$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$iw - z = 0$$
$$w + iz = 0$$

$$\cancel{AZ} \ z = iw \qquad (w, z) = (w, iw)$$
$$w = -iz$$

5.10 Linearly Independent eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are the corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent.

Proof: Suppose by contradiction that $v_1, \ldots, v_m$ is linearly dependent. By the Linear Dependence Lemma (2.21 of Axler), there exists the smallest positive integer $k \in \{1, \ldots, m\}$ such that

$$v_k \in \text{span}(v_1, \ldots, v_{k-1}).$$

So there exist $a_1, \ldots, a_{k-1} \in \mathbb{F}$ such that

$$v_k = a_1 v_1 + \ldots + a_{k-1} v_{k-1}$$

So we have

$$\begin{aligned} Tv_k &= T(a_1 v_1 + \ldots + a_{k-1} v_{k-1}) \\ &= a_1 T v_1 + \ldots + a_{k-1} T v_{k-1} \\ &= a_1 (\lambda_1 v_1) + \ldots + a_{k-1} (\lambda_{k-1} v_{k-1}) \\ \bigstar &= a_1 \lambda_1 v_1 + \ldots + a_{k-1} \lambda_{k-1} v_{k-1} \end{aligned}$$

Multiply by λ_k both sides of $v_k = a_1 v_1 + \ldots + a_{k-1} v_{k-1}$

to write $\boxed{Tv_k = \lambda_k v_k} = a_1 \lambda_k v_1 + \ldots + a_{k-1} \lambda_k v_{k-1}$ $\bigstar\bigstar$

b/c v_k is an eigenvector of T

So we also get

$$\begin{aligned} 0 &= \overset{\bigstar - \bigstar\bigstar}{Tv_k - Tv_k} \\ &= (a_1 \lambda_1 v_1 + \ldots + a_{k-1} \lambda_{k-1} v_{k-1}) - (a_1 \lambda_k v_1 + \ldots + a_{k-1} \lambda_k v_{k-1}) \\ &= a_1 (\lambda_1 - \lambda_k) v_1 + \ldots + a_{k-1} (\lambda_{k-1} - \lambda_k) v_{k-1} \end{aligned}$$

$\lambda_1 \neq \lambda_k$ $\qquad$ $\lambda_{k-1} \neq \lambda_k$ b/c we assume eigenvalues are distinct.

Since k is the smallest ~~positive~~ positive integer for which $v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$, it follows that $v_1, \dots, v_{k-1}$ is linearly independent. So we get $a_1(\lambda_1 - \lambda_k) = 0, \dots, a_{k-1}(\lambda_{k-1} - \lambda_k) = 0$.

But $\lambda_1, \dots, \lambda_m$ are distinct, so $a_1 = 0, \dots, a_{k-1} = 0$

So each v_k is zero, which contradicts our hypothesis that v_k is an eigenvector.

So $v_1, \dots, v_m$ is linearly independent $\blacksquare$