7/16/2019 Extra Credit Lecture: Chapter 5 (MATH 132)

Eigenvalues, Eigenvectors, and Invariant Subspaces

5A Invariant subspaces

Defn 5.2 Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called invariant under T if $u \in U$ implies $Tu \in U$.

Eg 5.3

- $\{0\}$: If $u \in \{0\}$, then $u = 0$. So we have
$$Tu = T(0) = 0 \in \{0\}.$$
So $\{0\}$ is invariant under T.
- V: If $u \in V$, then $Tu \in V$. So V is invariant under T.
- null T: If $u \in \operatorname{null} T$, then $Tu = 0$. So we have
$$T(Tu) = T(0) = 0$$
So $Tu \in \operatorname{null} T$, and so null T is invariant under T.
- range T: If $u \in \operatorname{range} T$, then $u = Tv$ for some $v \in V$. So we have $Tu = T(Tv)$. Since $Tv \in V$, we get $Tu \in \operatorname{range} T$. So range T is invariant under T.

Eigenvalues and Eiegenvectors

Defn 5.5 Suppose $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is called an eigenvalue of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$. Such a vector $v \in V$ is called an eigenvector of T corresponding to the eigenvalue λ.

5.6 Equivalent conditions to be an eigenvalue

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

(a): λ is an eigenvalue of T
(b): $T - \lambda I$ is not injective
(c): $T - \lambda I$ is not surjective
(d): $T - \lambda I$ is not invertible

proof: (a) is equivalent to (b): ((a) implies (b), and (b) implies (a))

The equation $Tv = \lambda v$ is equivalent to the equation $(T - \lambda I)v = 0$.

$Tv = \lambda v$, iff $Tv - \lambda v = 0$, iff $Tv - \lambda I v = 0$, iff $(T - \lambda I)v = 0$

Since $v \neq 0$, we conclude that $T - \lambda I$ is not injective, b: $T - \lambda I$ is not injective.

(b) implies (a): conversely, since $Tv = \lambda v$, so a: λ is an equivalent of T

(b), (c), (d) are equivalent to each other

By 3.62 of Axler the following are equivalent:

$T - \lambda I$ is not invertible,

$T - \lambda I$ is not injective,

and $T - \lambda I$ is not surjective.

Defn 5.7 Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an eigenvector of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Eg 5.8 Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w, z) = (-z, w)$

Find the eigenvalues and eigenvectors of T.

Soln: Let $\lambda \in \mathbb{F}$ satisfy $T(w,z) = \lambda(w,z)$.

Since $T(w,z) = (-z, w)$, we obtain

$$\begin{aligned}(-z, w) &= \lambda(w, z)\\ &= (\lambda w, \lambda z).\end{aligned}$$

From $(-z, w) = (\lambda w, \lambda z)$, we get a system of equations

$$-z = \lambda w$$
$$w = \lambda z$$

Substitute $w = \lambda z$ into $-z = \lambda w$, we get

$$\begin{aligned}-z &= \lambda w\\ &= \lambda(\lambda z)\\ &= \lambda^2 z.\end{aligned}$$

If $z = 0$, then $w = 0$, so $(w,z) = (0,0)$. If (w,z) is an eigenvector of T, then $z \neq 0$.

So $-z = \lambda^2 z$ implies $-1 = \lambda^2$ or $\lambda = i$, $\lambda = -i$

so $i, -i$ are eigenvalues of T.

$\lambda = i$

$T(w,z) = i(w,z)$

implies $(-z, w) = (iw, iz)$

so $-z = iw$

$w = iz$.

$z = -iw$

Therefore,

$(w,z) = (w, -iw)$

is an eigenvector of T corresponding to $\lambda = i$ for all $w \in \mathbb{C}$.

$\lambda = -i$

$T(w,z) = -i(w,z)$

implies $(-z, w) = (-iw, -iz)$

so $-z = -iw$,

$w = -iz$

$z = iw$

Therefore,

$(w,z) = (w, iw)$

is an eigenvector of T corresponding to $\lambda = -i$ for all $w \in \mathbb{C}$.

Alternate Method for Eg 5.8

Find $M(T) = M(T, \text{standard basis of } \mathbb{C}^2, \text{standard basis of } \mathbb{C}^2)$

Find $a, b, c, d \in \mathbb{F}$ that satisfy

So $M(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$M(T) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

$$\begin{pmatrix} aw + bz \\ cw + dz \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

$a = 0, \quad b = -1, \quad c = 1, d = 0$

$$\begin{pmatrix} 0w + (-1)z \\ (1)w + 0z \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

$$M(T - \lambda I) = M(T) - \lambda M(I)$$
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$

$\det(M(T - \lambda I)) = 0$

$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$

$\lambda^2 + 1 = 0$

$\lambda = i, \ \lambda = -i$

$M(T - iI)(w, z) = (0, 0)$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-iw - z = 0$

$w - iz = 0$

$-z = iw$

$w = iz$

$z = -iw$

$(w, z) = (w, -iw)$

$M(T - (-i)I)(w, z) = (0, v)$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$iw - z = 0$

$w + iz = 0$

$z = iw$

$w = -iz$

$(w, z) = (w, iw)$

5.10 Linearly Independent eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are the corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent

proof: Suppose by contradiction that $v_1, \ldots, v_m$ is linearly dependent. By the Linear Dependence Lemma (2.21 of Axler), there exists the smallest positive integer $k \in \{1, \ldots, m\}$ such that

$v_k \in \operatorname{span}(v_1, \cdots, v_{k-1})$

so there exist $a_1, \cdots, a_{k-1} \in \mathbb{F}$ such that

$$v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}.$$

So we have

$$\begin{aligned} Tv_k &= T(a_1 v_1 + \cdots + a_{k-1} v_{k-1}) \\ &= a_1 Tv_1 + \cdots + a_{k-1} Tv_{k-1} \\ &= a_1(\lambda v_1) + \cdots + a_{k-1}(\lambda v_{k-1}) = \star\, a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1} \end{aligned}$$

Multiply by λ_k both sides of $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$ to write $\star\star$ $\boxed{Tv_k = \lambda_k v_k} = a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1}$

because v_k is an eigenvector of T.

So we get $\star - \star\star$

$$\begin{aligned} 0 &= Tv_k - Tv_k \\ &= (a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}) - (a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1}) \\ &= a_1 (\overset{\lambda_1 \neq \lambda_k}{\lambda_1 - \lambda_k}) v_1 + \cdots + a_{k-1} (\overset{\lambda_{k-1} \neq \lambda_k}{\lambda_{k-1} - \lambda_k}) v_{k-1}. \end{aligned}$$

b/c we assumed the eigen-values are are distinct

Since k is the smallest positive integer for which $v_k \in \operatorname{span}(v_1, \cdots, v_{k-1})$, it follows that $v_1, \cdots, v_{k-1}$ is linearly independent. So we get

$$a_1(\lambda_1 - \lambda_k) = 0, \cdots, a_{k-1}(\lambda_{k-1} - \lambda_k) = 0$$

But $\lambda_1, \cdots, \lambda_m$ are distinct. So

$$a_1 = 0, \cdots, a_{k-1} = 0$$

So each v_k is zero, which contradicts one hypothesis that v_k is an eigenvector.

So $v_1, \cdots, v_m$ is linearly independent. $\boxtimes$