07-16-19

EXTRA CREDIT LECTURE CHAPTER 5

Eigenvalues, Eigenvectors, and Invariant Subspaces

5A Invariant Subspaces

5.2 Definition

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called **invariant** under T, if $u \in U$ implies $Tu \in U$

5.3 Example

- $\{0\}$: If $u \in \{0\}$, then $u = 0$. So we have

$$\begin{aligned} Tu &= T(0) \\ &= 0 \quad \in \{0\} \end{aligned}$$

so $\{0\}$ is invariant under T.

- V: If $u \in V$, then $Tu \in V$

So V is invariant under T

- null T: If $u \in \text{null } T$, then $Tu = 0$. So we have

$$\begin{aligned} T(Tu) &= T(0) \\ &= 0 \end{aligned}$$

So $Tu \in \text{null } T$, and so null T is invariant under T.

- range T: If $u \in \text{range } T$, then $u = Tv$ for some $v \in V$

So we have

$$Tu = T(Tv)$$

Since $Tv \in V$, we get $Tu \in \text{range } T$.

So range T is invariant under T.

Eigenvalues and Eigenvectors

5.5 Definition

- Suppose $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.
- Such a vector $v \in V$ is called an **eigenvector** of T corresponding to the eigenvalue λ.

5.6 Equivalent conditions to be an eigenvalue

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$

Then the following are equivalent:

(a): λ is an eigenvalue of T

(b): $T - \lambda I$ is not injective

(c): $T - \lambda I$ is not surjective

(d): $T - \lambda I$ is not invertible

$Tv = \lambda v$

iff $Tv - \lambda v = 0$

iff $Tv - \lambda I v = 0$

iff $(T - \lambda I)(v) = 0$

Proof: (a) is equivalent to (b): ((a) implies (b), and (b) implies (a))

(a) implies (b) The equation: $Tv = \lambda v$ is equivalent to the equation

$$(T - \lambda I)v = 0 \quad \text{(b)}$$

Since $v \neq 0$, we conclude that $T - \lambda I$ is not injective.

(b) implies (a) converse by. Since $T - \lambda I$ is not injective, $Tv = \lambda v$ so λ is an eigenvalue of T (a)

(b), (c), (d) are equivalent to each other

By 3.69 of Axler, the following are equivalent:

$T - \lambda I$ is not invertible

$T - \lambda I$ is not injective

and $T - \lambda I$ is not surjective

5.7 Definition

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an eigenvector of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$

5.8 Example

Define $T \in \mathcal{L}(\mathbb{C}^2)$ by

$$T(w, z) = (z, w)$$

Find the eigenvalues and eigenvectors of T

Solution: Let $\lambda \in \mathbb{F}$ satisfy

$$T(w, z) = \lambda(w, z)$$

Since $T(w, z) = (-z, w)$, we obtain

$$(-z, w) = \lambda(w, z) = (\lambda w, \lambda z)$$

From $(-z, w) = (\lambda w, \lambda z)$, we get a system of equations

$$-z = \lambda w$$
$$w = \lambda z$$

Substitute $w = \lambda z$ into $-z = \lambda w$, we get

$$-z = \lambda w = \lambda(\lambda z) = \lambda^2 z$$

If $z = 0$, then $w = 0$, so $(w, z) = (0, 0)$. If (w, z) is an eigenvector of T, then $z \neq 0$

so $-z = \lambda^2 z$ implies $-1 = \lambda^2$, or

$$\lambda = i, \quad \lambda = -i$$

so $i, -i$ are our eigenvalues of T

$\lambda = i$ $T(w, z) = i(w, z)$ implies $(-z, w) = (iw, iz)$ so $-z = iw$ $w = iz$ Therefore $(w, z) = (w, -iw)$ is an eigenvector of T for $\lambda = i$, for all $w \in \mathbb{C}$	$\lambda = -i$ $T(w, z) = i(w, z)$ implies $(-z, w) = (-iw, iz)$ so $-z = -iw$ $w = -iz$ Therefore $(w, z) = (w, iw)$ is an eigenvector of T corresponding to $\lambda = -i$ for all $w \in \mathbb{C}$

Alternate method: Find

$$M(T) = M\left(T, \begin{matrix}\text{standard basis} \\ \text{of } \mathbb{C}^2\end{matrix}, \begin{matrix}\text{standard basis} \\ \text{of } \mathbb{C}^2\end{matrix}\right)$$

so $M(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$M(T) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Find $a, b, c, d \in \mathbb{F}$ that satisfy

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

$$\begin{pmatrix} aw + bz \\ cw + dz \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

$a = 0, \quad b = -1$

$c = 1, \quad d = 0$

$$\begin{pmatrix} 0w + (-1)z \\ 1w + 0z \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

$M(T - (-i)I)(w, z) = (0, 0)$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$iw - z = 0$

$w + iz = 0$

$z = iw$

$w = -iz$

$(w, z) = (w, iw)$

$M(T - \lambda I) = M(T) - \lambda M(I)$

$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$

$\det(M(T - \lambda I)) = 0$

$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$

$\lambda^2 + 1 = 0$

$\lambda = i, \ \lambda = -i$

$M(T - iI)(w, z) = (0, 0)$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-iw - z = 0$

$w + z = 0$

$-z = iw$

$w = iz$

$(w, z) = (w, -iw)$

5.10 Linearly Independent eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigevalues of T and $v_1, \dots, v_m$ are the corresponding eigenvectors. Then $v_1, \dots, v_m$ is linearly independent.

Proof: Suppose by contradiction that $v_1, \dots, v_m$ is linearly dependent. By the Linear Dependence Lemma (2.21 of Axler), there exists the smallest positive integer $k \in \{1, \dots, m\}$

such that

$$v_k \in \operatorname{span}(v_1, \dots, v_{k-1})$$

So there exist $a_1, \dots, a_{k-1} \in \mathbb{F}$ such that

$$v_k = a_1 v_1 + \dots + a_{k-1} v_{k-1}$$

So we have

$$Tv_k = T(a_1 v_1 + \dots + a_{k-1} v_{k-1})$$

$$= a_1 Tv_1 + \dots + a_{k-1} Tv_{k-1}$$

$$= a_1(\lambda v_1) + \dots + a_{k-1}(\lambda v_{k-1})$$

$$\star \quad = a_1 \lambda_1 v_1 + \dots + a_{k-1} \lambda_{k-1} v_{k-1}$$

Multiply by λ_k both sides of $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$ to write

$$** \quad \boxed{Tv_k = \lambda_k v_k} = a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1}$$

v_k is an eigenvector of T

so we get

$$\begin{aligned} 0 &= Tv_k - Tv_k \\ &= (a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}) - (a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1}) \\ &= a_1 \underbrace{(\lambda_1 - \lambda_k)}_{\lambda_1 \neq \lambda_k} v_1 + \cdots + a_{k-1} \underbrace{(\lambda_{k-1} - \lambda_k)}_{\lambda_{k-1} \neq \lambda_k} v_{k-1} \end{aligned}$$

we assumed that eigenvalues are distinct

Since k is the smallest positive integer for which $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$, it follows that $v_1, \ldots, v_{k-1}$ is linearly independent. So we get

$$a_1(\lambda_1 - \lambda_k) = 0, \ldots, a_{k-1}(\lambda_{k-1} - \lambda_k) = 0$$

But $\lambda_1, \ldots, \lambda_m$ are distinct. So

$$a_1 = 0, \ldots, a_{k-1} = 0$$

So each v_k is zero, which contradicts our hypothesis that v_k is an eigenvector

So $v_1, \ldots, v_m$ is linearly independent

$\boxtimes$