Extra Credit Lecture: Chapter 5

Eigenvalues, Eigenvectors, and Invariant Subspaces

5.A Invariant Subspaces

5.2 Definition

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called invariant under T if $u \in U$ implies $Tu \in U$.

5.3 Example

- $\{0\}$: If $u \in \{0\}$, then $u = 0$. So we have

$$\begin{aligned} Tu &= T(0) \\ &= 0 \\ &\in \{0\}. \end{aligned}$$

So $\{0\}$ is invariant under T.

- V: If $u \in V$, then $Tu \in V$.

So V is invariant under T.

- null T: If $u \in$ null T, then $Tu = 0$. So we have

$$\begin{aligned} T(Tu) &= T(0) \\ &= 0 \end{aligned}$$

So $Tu \in$ null T, and so null T is invariant under T.

- range T: If $u \in$ range T, then $u = Tv$ for some $v \in V$

So we have $Tu = T(Tv)$

Since $Tv \in V$, we get $Tu \in$ range T.

So range T is invariant under T.

5.5 Definition

Suppose $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is called an <u>eigenvalue</u> of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Such a vector $v \in V$ is called an <u>eigenvector</u> of T corresponding to the eigenvalue λ.

5.6 Equivalent conditions to be an eigenvalue

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$.

Then the following are equivalent:

(a): λ is an eigenvalue of T

(b): $T - \lambda I$ is not injective

(c): $T - \lambda I$ is not surjective

(d): $T - \lambda I$ is not invertible.

(b) implies (a):

Conversely, since $T - \lambda T$ is not injective, $Tv = \lambda v$.

So λ is an eigenvalue of T.

(a)

<u>Proof</u>: <u>(a) is equivalent to (b)</u>: <u>(a) implies (b), and (b) implies (a)</u>

(a) implies (b): The equation $Tv = \lambda v$ is equivalent to the equation

$(T - \lambda I)v = 0$. Since $v \neq 0$, we conclude that $T - \lambda I$ is not injective.

(b)

$Tv = \lambda v$,

if and only if $Tv - \lambda v = 0$

if and only if $Tv - \lambda I v = 0$

if and only if $(T - \lambda I)(v) = 0$

<u>(b) (c), (d) are equivalent to each other</u>

By 3.69 of Axler, the following are equivalent:

$T - \lambda I$ is not invertible,

$T - \lambda I$ is not injective,

and $T - \lambda I$ is not surjective.

5.7 Definition

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T.

A vector $v \in V$ is called an <u>eigenvector</u> of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

5-8 Example

Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w, z) = (-z, w)$.

Find the eigenvalues and eigenvectors of T.

Solution: Let $\lambda \in \mathbb{F}$ satisfy $T(w, z) = \lambda(w, z)$.

Since $T(w, z) = (-z, w)$, we obtain

$$(-z, w) = \lambda(w, z) = (\lambda w, \lambda z).$$

From $(-z, w) = (\lambda w, \lambda z)$, we get a system of equations

$$-z = \lambda w$$
$$w = \lambda z$$

Substitute $w = \lambda z$ into $-z = \lambda w$, we get

$$-z = \lambda w = \lambda(\lambda z) = \lambda^2 z.$$

If $z = 0$, then $w = 0$, so $(w, z) = (0, 0)$. If (w, z) is an eigenvector of T, then $z = 0$.

So $-z = \lambda^2 z$ implies $-1 = \lambda^2$, or $\lambda = i$, $\lambda = -i$.

So $i, -i$ are our eigenvalues of T.

$\lambda = i$: $T(w, z) = i(w, z)$

implies

$(-z, w) = (iw, iz)$

So

$-z = iw$,

$w = iz$.

$z = -iw$

Therefore,

$(w, z) = (w, -iw)$

is an eigenvector of T corresponding to $\lambda = i$ for all $w \in \mathbb{C}$.

$\lambda = -i$: $T(w, z) = -i(w, z)$

implies

$(-z, w) = (-iw, -iz)$

So

$-z = iw$.

$w = -iz$.

Therefore, $z = iw$.

$(w, z) = (w, iw)$

is an eigenvector of T corresponding to $\lambda = -i$ for all $w \in \mathbb{C}$.

Alternate method: Find

$$M(T) = M(T, \text{ standard basis of } \mathbb{C}^2, \text{ standard basis of } \mathbb{C}^2)$$

Find $a, b, c, d \in \mathbb{F}$ that satisfy

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

$$\begin{pmatrix} aw + bz \\ cw + dz \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

$a = 0$, $b = -1$

$c = 1$, $d = 0$

$$\begin{pmatrix} 0w + (-1)z \\ 1w + 0z \end{pmatrix} = \begin{pmatrix} -z \\ w \end{pmatrix}$$

so $M(T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$M(T) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$M(T - \lambda I) = M(T) - \lambda M(I)$

$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$

$\det(M(T - \lambda I)) = 0$

$\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$

$\lambda^2 + 1 = 0$

$\lambda = i, \lambda = -i$

$M(T - iI)(w, z) = (0, 0)$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-iw - z = 0$

$w - iz = 0$

$-z = iw$

$w = iz$ $\quad z = -iw$

$(w, z) = (w, -iw)$

$M(T - (-i)I)(w, z) = (0, 0)$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$iw - z = 0$

$w + iz = 0$

$z = iw$

$w = -iz$

$(w, z) = (w, iw)$

5.10 Linearly independent eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are the corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent.

Proof: Suppose by contradiction that $v_1, \ldots, v_m$ is linearly dependent. By the Linear Dependence Lemma (2.21 of Axler), there exists the smallest positive integer $k \in \{1, \ldots, m\}$ such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$.

So there exist $a_1, \ldots, a_{k-1} \in \mathbb{F}$ such that $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$.

So we have $Tv_k = T(a_1 v_1 + \cdots + a_{k-1} v_{k-1})$ $\cancel{= a_1 \lambda v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}}$

$= a_1 T v_1 + \cdots + a_{k-1} T v_{k-1}$ $\quad \star = a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}$

$= a_1(\lambda_1 v_1) + \cdots + a_{k-1}(\lambda_{k-1} v_{k-1})$

Multiply by λ_k both sides of $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$ to write

$$☆☆ \quad \boxed{T v_k = \lambda_k v_k} = a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1}$$

because v_k is an eigenvector of T.

So we get
$$0 = \overset{☆}{T v_k} - \overset{☆☆}{T v_k}$$
$$= (a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}) - (a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1})$$
$$= a_1 \underset{}{(\lambda_1 - \lambda_k)} v_1 + \cdots + a_{k-1} (\lambda_{k-1} - \lambda_k) v_{k-1}$$

$\lambda_1 \neq \lambda_k$

$\lambda_{k-1} \neq \lambda_k$ b/c we ~~…~~ assume the eigenvalues are distinct.

Since k is the smallest positive integer for which $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$ it follows that $v_1, \ldots, v_{k-1}$ is linearly independent. So we get

$$a_1(\lambda_1 - \lambda_k) = 0, \ldots, a_{k-1}(\lambda_{k-1} - \lambda_k) = 0.$$

But $\lambda_1, \ldots, \lambda_m$ are distinct. So $a_1 = 0, \ldots, a_{k-1} = 0$.

So each v_k is zero, which contradicts our hypothesis that v_k is an eigenvector.

So $v_1, \ldots, v_m$ is linearly independent.