MATH 131: Linear Algebra I
University of California, Riverside
Quiz 1 Solutions
June 26, 2019

(10pts) 1. This question will ask you to recall your knowledge of quantifiers that we covered in our discussion today.

(4pts) a. Write the phrases and the informal symbols of both the universal quantifier and the existential quantifier.

Answers. The following phrases that describe the universal and existential quantifiers include, but are not limited to, the following.

- Universal quantifier: For all ($\forall$), for any, for every, for each.
- Existential quantifier: There exists ($\exists$), there is, for some, for one.

(6pts) b. Write 6 different sentences of your own that incorporates both the universal and existential quantifiers in the same sentence. (Do not repeat any of the examples we have already done together in discussion before this quiz; those sentences will give you no credit.)

Answers. Here are six random (and not-so-interesting) examples.

- For all $\epsilon > 0$, there exists $\delta > 0$ such that, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$ for all $x, c \in \mathbb{R}$ with c fixed.
- For all television programs on NBC in 2019, there exists the Ellen show.
- For each student in the MATH 131 class at UCR in Summer 2019, there is an instructor teaching that class.
- For every apple, there are some seeds near the center of its core.
- For every hour that passes, there is an ever-increasing appetite in your stomach.
- For all the people on Earth, there exists a common ancestor a long time ago.

(10pts) 2. Let a, b be even integers.

(4pts) a. Prove that $a + b, a - b, ab$ are also even integers. Use a counterexample to show that $\frac{a}{b}$ (assuming $b \neq 0$) does not have to be an even integer.

Proof. Since a, b are even integers, there exist integers k, l that satisfy $a = 2k$ and $b = 2l$. So we have

$$\begin{aligned} a + b &= 2k + 2l = 2(k + l), \\ a - b &= 2k - 2l = 2(k - l), \\ ab &= (2k)(2l) = 2(2kl). \end{aligned}$$

Since $k + l, k - l, 2kl$ are also integers, we conclude that $a + b, a - b, ab$ are even integers. □

(6pts) b. Prove that $3a^2 - 4b - 5$ is an odd integer.

Proof. Again, since a, b are even integers, there exist integers k, l that satisfy $a = 2k$ and $b = 2l$. So we have

$$\begin{aligned} 3a^2 - 4b - 5 &= 3(2k)^2 - 4(2l) - 5 \\ &= 12k^2 - 8l - 5 \\ &= 12k^2 - 8l - 6 + 1 \\ &= 2(6k^2 - 4l - 3) + 1. \end{aligned}$$

Since $6k^2 - 4l - 3$ is also an integer, we conclude that $3a^2 - 4b - 5$ is an odd integer. □

(10pts) 3. Consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by the function $f(x) = x^3 - 6x^2 + 9x$.

(5pts) a. Prove that we have $f(x) \geq 0$ for all $x \geq 0$.

Proof. We have

$$\begin{aligned} f(x) &= x^3 - 6x^2 + 9x \\ &= x(x^2 - 6x + 9) \\ &= x(x - 3)^2. \end{aligned}$$

Since the square of any real number must be non-negative, we know that $(x - 3)^2 \geq 0$ is true. Since we also assumed $x \geq 0$, we conclude

$$\begin{aligned} f(x) &= x(x - 3)^2 \\ &\geq 0, \end{aligned}$$

as desired. □

(3pts) b. Use a counterexample to show that the statement $f(x) \geq 0$ for all $x \in \mathbb{R}$ is false.

Proof. Let $x = -1 \in \mathbb{R}$. Then we have

$$\begin{aligned} f(-1) &= (-1)^3 - 6(-1)^3 + 9(-1) \\ &= -16 \\ &< 0. \end{aligned}$$

So the statement $f(-1) \geq 0$ is false, which means the statement $f(x) \geq 0$ for all $x \in \mathbb{R}$ is false. □

(2pts) c. Define a new function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = -f(x)$. Is it true that we have $g(x) \leq 0$ for all $x \geq 0$? Either prove that this statement is true or give a counterexample to show that this statement is false.

Proof. This is true, and we will prove the statement. By part (a), we have $f(x) \geq 0$ for all $x \geq 0$. So we have

$$\begin{aligned} g(x) &= -f(x) \\ &\leq 0 \end{aligned}$$

for all $x \geq 0$. □

(10pts) 4. Suppose $U_1, \ldots, U_m$ are subspaces of V. Prove that $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Proof (1.39 of Axler). First, we will prove that $U_1 + \cdots + U_m$ is a subspace of V. Since we assumed that $U_1, \ldots, U_m$ are subspaces of V, they satisfy:

- Additive identities: $0 \in U_1, \ldots, 0 \in U_m$;
- Closed under addition: $u_1, w_1 \in U_1, \ldots, u_m, w_m \in U_m$ imply $u_1 + w_1 \in U_1, \ldots, u_m + w_m \in U_m$;
- Closed under scalar multiplication: for all $\lambda \in \mathbb{F}$, we have $u_1 \in U_1, \ldots, u_m \in U_m$ imply $\lambda u_1 \in U_1, \ldots, \lambda u_m \in U_m$.

From these properties of $U_1, \ldots, U_m$, we will show that $U_1 + \cdots + U_m$ also has the same properties. We have:

- Additive identity: We have $0 = 0 + \cdots + 0 \in U_1 + \cdots + U_m$.
- Closed under addition: Suppose we have $u, w \in U_1 + \cdots + U_m$. Then we can write $u = u_1 + \cdots + u_m$ and $w = w_1 + \cdots + w_m$ for some $u_1, w_1 \in U_1, \ldots, u_m, w_m \in U_m$. So the sum of u, w is

$$\begin{aligned} u + w &= (u_1 + \cdots + u_m) + (w_1 + \cdots + w_m) \\ &= (u_1 + w_1) + \cdots + (u_m + w_m) \\ &\in U_1 + \cdots + U_m. \end{aligned}$$

- Closed under scalar multiplication: For all $\lambda \in \mathbb{F}$, we have

$$\begin{aligned} \lambda u &= \lambda(u_1 + \cdots + u_m) \\ &= \lambda u_1 + \cdots + \lambda u_m \\ &\in U_1 + \cdots + U_m. \end{aligned}$$

This completes the proof that $U_1 + \cdots + U_m$ is a subspace of V. Next, we will show that $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$. Suppose we have $u_i \in U_i$ for all $i = 1, \ldots, m$. Then we can write

$$\begin{aligned} u_i &= 0 + \cdots + 0 + u_i + 0 + \cdots + 0 \\ &\in U_1 + \cdots + U_{i-1} + U_i + U_{i+1} + \cdots + U_m \\ &= U_1 + \cdots + U_m. \end{aligned}$$

So we have $U_i \subset U_1 + \cdots + U_m$ for all $i = 1, \ldots, m$; in other words, we have $U_1, \ldots, U_m \subset U_1 + \cdots + U_m$. Furthermore, due to the property of closed under addition for subspaces, every subspace of V that contains $U_1, \ldots, U_m$ must also contains finite sums of elements of $U_1, \ldots, U_m$. This means in particular that every subspace that contains $U_1, \ldots, U_m$ must also contain $U_1 + \cdots + U_m$. In other words, $U_1 + \cdots + U_m$ is contained in every subspace of V that also contains $U_1, \ldots, U_m$. Therefore, $U_1 + \cdots + U_m$ is the smallest subspace of V that contains $U_1, \ldots, U_m$. □

(10pts) 5. Suppose U and W are subspaces of a vector space V. Prove that $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof (1.45 of Axler). Forward direction: If $U + W$ is a direct sum, then $U \cap W = \{0\}$. Suppose that $U + W$ is a direct sum and that we have $v \in U \cap W$. Then we have $v \in U$ and $v \in W$. Furthermore, since W is a subspace of V, it follows that W is closed under scalar multiplication, which means in particular we have $-v \in W$. So we can write $0 = v + (-v) \in U + W$. However, since we assumed that $U + W$ is a direct sum, the representation of 0 is unique. According to 1.44 of Axler, the only way to 0 as a sum of v and $-v$ is to take v and $-v$ equal to 0; in other words, we must have $v = 0$. So we arrive at $U \cap W \subset \{0\}$. On the other hand, we also have $0 \in U$ and $0 \in W$, which means we have $0 \in U \cap W$, or $\{0\} \subset U \cap W$. Therefore, we have the set equality $U \cap W = \{0\}$.

Backward direction: If $U \cap W = \{0\}$, then $U + W$ is a direct sum. Suppose that we have $U \cap W = \{0\}$. Let $u \in U, w \in W$ satisfy $0 = u + w$. Since W is a subspace of V, it follows that W is closed under scalar multiplication, which means in particular with $0 = u + w$ we have $u = -w \in W$. So we have $u \in U$ and $u \in W$, which means we have $u \in U \cap W$. Sine $U \cap W = \{0\}$, we must have $u = 0$. Furthermore, $0 = u + w$ with $u = 0$ implies $w = 0$. So $u = 0$ and $w = 0$, from which 1.44 of Axler asserts that $U + W$ is a direct sum. □