MATH 131: Linear Algebra I
University of California, Riverside
Quiz 2 Solutions
July 10, 2019

(10pts) 1. Use contradiction to prove that, if $9n^3 + 7n^2 + 5n$ is even, then n is even.

Proof. Suppose by contradiction that n is not even' that is, suppose n is odd. Then there exists some integer k that satisfies $n = 2k + 1$. But then we have

$$\begin{aligned} 9n^3 + 7n^2 + 5n &= 9(2k+1)^3 + 7(2k+1)^2 + 5(2k+1) \\ &= 9(8k^3 + 12k^2 + 6k + 1) + 7(4k^2 + 4k + 1) + 5(2k+1) \\ &= (72k^3 + 108k^2 + 54k + 9) + (28k^2 + 28k + 7) + (10k + 5) \\ &= 72k^3 + 136k^2 + 92k + 21 \\ &= 72k^3 + 136k^2 + 92k + 20 + 1 \\ &= 2(36k^3 + 68k^2 + 46k + 10) + 1. \end{aligned}$$

Since $36k^3 + 68k^2 + 46k + 10$ is also an integer, we conclude that $9n^3 + 7n^2 + 5n$ is odd. But this contradicts our assumption that n^3 is even. □

(10pts) 2. Use contradiction to prove that $\sqrt{3}$ is irrational.

Proof. Suppose by contradiction that $\sqrt{3}$ is rational. Then there exist integers a, b that satisfy

$$\sqrt{3} = \frac{a}{b},$$

where $\frac{a}{b}$ is a fraction expressed in lower terms, meaning that a and b do not have a common factor greater than 1. Squaring both sides of the above equation, we get

$$3 = \frac{a^2}{b^2},$$

from which we can algebraically rewrite as

$$3b^2 = a^2.$$

Since $3b^2$ is a multiple of 3, it follows that a^2 is a multiple of 3. Furthermore, we claim that, if a is an integer such that a^2 is a multiple of 3, then a is a multiple of 3. To prove this claim, suppose by contradiction that a is not a multiple of 3. Then there exists an integer k such that we have either $a = 3k + 1$ or $a = 3k + 2$. At this point, we will continue our argument by breaking down into separate cases.

- Case 1: If $a = 3k + 1$, then

$$\begin{aligned} a^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1, \end{aligned}$$

  and so a^2 is not a multiple of 3, which contradicts our earlier result saying that a^2 is a multiple of 3.
- Case 2: If $a = 3k + 2$, then

$$\begin{aligned} a^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k + 1) + 1, \end{aligned}$$

  and so a^2 is not a multiple of 3, which contradicts our earlier result saying that a^2 is a multiple of 3.

Therefore, we must have $a = 3k$; that is, a is a multiple of 3, proving our claim. Now, if we continue to assume that a is a multiple of 3, then we still write $a = 3k$ for some integer k. This implies that we have

$$\begin{aligned} 3b^2 &= a^2 \\ &= (3k)^2 \\ &= 9k^2, \end{aligned}$$

from which we divide both sides by 3 to conclude

$$b^2 = 3k^2,$$

and so b^2 is a multiple of 3. Invoking our earlier claim that we proved already, we conclude that b is a multiple of 3. Therefore, a and b have a common factor of 3, which contradicts our earlier result stating that a and b do not have a common factor greater than 1. Therefore, $\sqrt{3}$ is irrational. □

(10pts) 3. Prove that a list $v_1, \ldots, v_n$ is a basis of V if and only if every $v \in V$ can be written uniquely in the form $v = a_1v_1 + \cdots + a_nv_n$, where $a_1, \ldots, a_n \in \mathbb{F}$.

Proof (2.29 of Axler). Forward direction: If a list $v_1, \ldots, v_n$ is a basis of V, then every $v \in V$ can be written uniquely in the form $v = a_1v_1 + \cdots + a_nv_n$. Suppose $v_1, \ldots, v_n$ is a basis of V. Then $v_1, \ldots, v_n$ is a linearly independent set that spans V. Since the list $v_1, \ldots, v_n$ spans V, there exist $a_1, \ldots, a_n \in \mathbb{F}$ that satisfy

$$v = a_1v_1 + \cdots + a_nv_n.$$

Next, we will show that this representation of v is unique. Suppose there exist $c_1, \ldots, c_n \in \mathbb{F}$ that satisfy

$$v = c_1v_1 + \cdots + c_nv_n.$$

Subtracting the two equations, we obtain

$$\begin{aligned} 0 &= v - v \\ &= (a_1v_1 + \cdots + a_nv_n) - (c_1v_1 + \cdots + c_nv_n) \\ &= (a_1 - c_1)v_1 + \cdots + (a_n - c_n)v_n. \end{aligned}$$

Since the list $v_1, \ldots, v_n$ is linearly independent, all the scalars are zero; that is, we have

$$a_1 - c_1 = 0, \ldots, a_n - c_n = 0,$$

or equivalently $a_1 = c_1, \ldots, a_n = c_n$, proving the uniqueness of the form of $v = a_1v_1 + \cdots + a_nv_n$.

Backward direction: If every $v \in V$ can be written uniquely in the form $v = a_1v_1 + \cdots + a_nv_n$, then $v_1, \ldots, v_n$ is a basis of V. First, we will prove that $v_1, \ldots, v_m$ spans V. Suppose there exist $a_1, \ldots, a_n \in \mathbb{F}$ such that we can write every $v \in V$ uniquely in the form

$$v = a_1v_1 + \cdots + a_nv_n.$$

Then v is a linear combination of $v_1, \ldots, v_m$, which means we have $v \in \text{span}(v_1, \ldots, v_m)$. So we have $V \subset \text{span}(v_1, \ldots, v_m)$. But $\text{span}(v_1, \ldots, v_m)$ is a subspace of V, according to 2.7 of Axler. So we have the set equality $V = \text{span}(v_1, \ldots, v_m)$, and so $v_1, \ldots, v_m$ spans V. Next, we will prove that $v_1, \ldots, v_n$ is linearly independent. Suppose $a_1, \ldots, a_n \in \mathbb{F}$ satisfy

$$a_1v_1 + \cdots + a_nv_n = 0.$$

We assumed that the form of every $v \in V$ is unique. In particular, there is a unique representation of $v = 0$. Therefore, the equation $a_1v_1 + \cdots + a_nv_n = 0$ implies

$$a_1 = 0, \ldots, a_m = 0,$$

and so $v_1, \ldots, v_n$ is linearly independent. Therefore, $v_1, \ldots, v_n$ is a basis of V. □

(10pts) 4. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Prove that there exists a unique linear map $T : V \to W$ such that $Tv_j = w_j$ for each $j = 1, \ldots, n$.

Proof (3.5 of Axler). First, we will show that there exists a linear map $T : V \to W$ such that $Tv_j = w_j$ for each $j = 1, \ldots, n$. Define $T : V \to W$ by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

for some $c_1, \ldots, c_n \in \mathbb{F}$. Since $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n$ is a basis of W, the representations $c_1v_1 + \cdots + c_nv_n \in V$ and $c_1w_1 + \cdots + c_nw_n \in W$ are unique. This means T indeed defines a function. Furthermore, if, for all $i, j = 1, \ldots, n$, we let

$$c_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

then T satisfies $Tv_j = w_j$. Next, we will prove that $T : V \to W$ is linear. Let $u, v \in V$ be arbitrary. Again, by 2.29 of Axler, the forms of every vector in V is unique, which means we can write

$$u = a_1v_1 + \cdots + a_nv_n$$

and

$$v = c_1v_1 + \cdots + c_nv_n$$

for some $a_1, \ldots, a_n, c_1, \ldots, c_n \in \mathbb{F}$. So, for all $\lambda \in \mathbb{F}$ and for all $u, v \in V$, we have

$$\begin{aligned} T(u + v) &= T((a_1v_1 + \cdots + a_nv_n) + (c_1v_1 + \cdots + c_nv_n)) \\ &= T((a_1 + c_1)v_1 + \cdots + (a_n + c_n)v_n) \\ &= (a_1 + c_1)w_1 + \cdots + (a_n + c_n)w_n \\ &= (a_1w_1 + \cdots + a_nw_n) + (c_1w_1 + \cdots + c_nw_n) \\ &= T(a_1v_1 + \cdots + a_nv_n) + T(c_1v_1 + \cdots + c_nv_n) \\ &= Tu + Tv \end{aligned}$$

and

$$\begin{aligned}
T(\lambda u) &= T(\lambda(a_1 v_1 + \cdots + a_n v_n)) \\
&= T((\lambda a_1)v_1 + \cdots + (\lambda a_n)v_n) \\
&= (\lambda a_1)w_1 + \cdots + (\lambda a_n)w_n \\
&= \lambda(a_1 w_1 + \cdots + a_n w_n) \\
&= \lambda T(a_1 u_1 + \cdots + a_n u_n) \\
&= \lambda T u.
\end{aligned}$$

So $T : V \to W$ is linear, or equivalently we have $T \in \mathcal{L}(V, W)$. Finally, we will prove that T is unique. Suppose $S \in \mathcal{L}(V, W)$ also satisfies

$$S v_j = w_j.$$

Then for all $v \in V$, we have

$$\begin{aligned}
Sv &= S(c_1 v_1 + \cdots + c_n v_n) \\
&= c_1 S v_1 + \cdots + c_n S v_n \\
&= c_1 w_1 + \cdots + c_n w_n \\
&= c_1 T v_1 + \cdots + c_n T v_n \\
&= T(c_1 v_1) + \cdots + T(c_n v_n) \\
&= T(c_1 v_1 + \cdots + c_n v_n) \\
&= Tv.
\end{aligned}$$

So $S = T$ on V, which proves the uniqueness of T. Therefore, we proved that $T : V \to W$ is a unique linear map that satisfies $T v_j = w_j$ for all $j = 1, \ldots, n$. □

(10pts) 5. Let V and W be vector spaces. If V is finite-dimensional and $T \in \mathcal{L}(V, W)$, prove that range T is also finite dimensional and we have $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.

Proof (3.22 of Axler). Let $u_1, \ldots, u_m$ be a basis of null T. Then we have $\dim \operatorname{null} T = m$ and $u_1, \ldots, u_m$ is a linearly independent list. By 2.33 of Axler, we can extend $u_1, \ldots, u_m$ to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V, which means $\dim V = m + n$. So the equation

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

becomes

$$m + n = m + \dim \operatorname{range} T.$$

So to complete this proof we need to prove that $Tv_1, \ldots, Tv_n$ is a basis of range T, which would establish that range T is finite-dimensional and $\dim \operatorname{range} T = n$. First, we will show that $Tv_1, \ldots, Tv_n$ spans range T. Let $v \in V$ be arbitrary. Since $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V, it spans V, which means we can write

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

for some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{F}$. Recall at the beginning of our proof that $u_1, \ldots, u_m$ is a basis of null T. Then we have $Tu_1 = 0, \ldots, Tu_m = 0$, and so we get

$$\begin{aligned}
Tv &= T(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) \\
&= a_1 T u_1 + \cdots + a_m T u_m + b_1 T v_1 + \cdots + b_n T v_n \\
&= a_1 \cdot 0 + \cdots + a_m \cdot 0 + b_1 T v_1 + \cdots + b_n T v_n \\
&= b_1 T v_1 + \cdots + b_n T v_n.
\end{aligned}$$

So Tv is a linear combination of $Tv_1, \ldots, Tv_n$, which means $Tv_1, \ldots, Tv_n$ spans range T. Next, we show $Tv_1, \ldots, Tv_n$ is linearly independent in range T. Suppose $c_1, \ldots, c_n \in \mathbb{F}$ satisfy

$$c_1 T v_1 + \cdots + c_n T v_n = 0.$$

Since T is linear, we have

$$\begin{aligned}
0 &= c_1 T v_1 + \cdots + c_n T v_n \\
&= T(c_1 v_1) + \cdots + T(c_n v_n) \\
&= T(c_1 v_1 + \cdots + c_n v_n),
\end{aligned}$$

and so we have $c_1 v_1 + \cdots + c_n v_n \in \operatorname{null} T$. Since $u_1, \ldots, u_m$ is a basis of null T, it spans null T, which means every vector in null T can be written as a linear combination of $u_1, \ldots, u_m$. In other words, since $c_1 v_1 + \cdots + c_n v_n$ is one such vector in null T, we can write

$$c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m.$$

Equivalently, we have

$$-d_1u_1 - \cdots - d_mu_m + c_1v_1 + \cdots + c_nv_n = 0.$$

Since $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V, it is linearly independent in V. So all the scalars are zero; that is, we have

$$-d_1 = 0, \ldots, -d_m = 0, c_1 = 0, \ldots, c_n = 0,$$

or equivalently,

$$d_1 = 0, \ldots, d_m = 0, c_1 = 0, \ldots, c_n = 0.$$

In particular, we have

$$c_1 = 0, \ldots, c_n = 0,$$

and so $Tv_1, \ldots, Tv_n$ is linearly independent in $\operatorname{range} T$. Therefore, $Tv_1, \ldots, Tv_n$ is a basis of $\operatorname{range} T$. So we conclude that $\operatorname{range} T$ is finite-dimensional with $\dim \operatorname{range} T = n$, and we conclude also

$$\begin{aligned}\dim V &= m + n \\ &= \dim \operatorname{null} T + \dim \operatorname{range} T,\end{aligned}$$

as desired. □