MATH 131: Linear Algebra I
University of California, Riverside
Quiz 3 Solutions
July 17, 2019

This quiz is open textbook and open lecture notes. The time limit is 40 minutes, unless the instructor grants a time extension.

(10pts) 1. State the contrapositive, converse, and inverse of the following statements in parts a-c. Do not forget to answer part d as well.

(3pts) a. If a real number is greater than 3, then its square is greater than 9.

Answers. The contrapositive, converse, and inverse are as follows:

- Contrapositive: If the square of a real number is not greater than 9, then the real number is not greater than 3.
- Converse: If the square of a real number is greater than 9, then the real number is greater than 3.
- Inverse: If a real number is not gerater than 3, then its square is not greater than 9.

(3pts) b. If a quadrilateral is a square, then it is a rectangle.

Answers. The contrapositive, converse, and inverse are as follows:

- Contrapositive: If a quadrilateral is not a rectangle, then it is not a square.
- Converse: If a quadrilateral is a rectangle, then it is a square.
- Inverse: If a quadrilateral is not a square, then it is not a rectangle.

(3pts) c. If Riverside did not exceed 100°F this month, then I would not have had to pay an expensive bill for the air conditioning in my house.

Answers. The contrapositive, converse, and inverse are as follows:

- Contrapositive: If I would have had to pay an expensive bill for the air conditioning in my house, then Riverside exceeded 100°F this month.
- Converse: If I would not have had had to pay an expensive bill for the air conditioning, then Riverside did not exceed 100°F this month.
- Inverse: If Riverside exceeded 100°F this month, then I would have had to pay an expensive bill for the air conditioning in my house.

(1pt) d. What is the key difference between a proving a contrapositive or using a proof by contradiction?

Answer. Proving by contrapositive requires assuming that the conclusion of a given statement is false and negating all the premises of that statement. On the other hand, proving by contradiction requires assuming that the conclusion is false, and finding a contradiction (any false statement), which includes but is not limited to negating at least one of the premises. Notice that proving by contrapositive is a special case of proving by contradiction.

(10pts) 2. Suppose n is a positive integer. Consider the statement: If $n^4 + 10n^2 + 21$ is not a multiple of 32, then n is even.

(1pt) a. State the contrapositive of the statement.

Contrapositive. If n is odd, then $n^4 + 10n^2 + 21$ is a multiple of 32. □

(9pts) b. Prove the contrapositive that you have written.

Proof. Since n is odd, there exists an integer k that satisfies $n = 2k + 1$. So we have

$$\begin{aligned} n^4 + 10n^2 + 21 &= (n^2 + 3)(n^2 + 7) \\ &= ((2k + 1)^2 + 3)((2k + 1)^2 + 7) \\ &= ((4k^2 + 4k + 1) + 3)((4k^2 + 4k + 1) + 7) \\ &= (4k^2 + 4k + 4)(4k^2 + 4k + 8) \\ &= (4(k^2 + k + 1))(4(k^2 + k + 2)) \\ &= 16(k^2 + k + 1)(k^2 + k + 2). \end{aligned}$$

We are left to prove that, for all integers k, one of $k^2 + k + 1$ and $k^2 + k + 2$ is even. We must split this proof into two possible cases.

- Case 1: Suppose $k^2 + k + 1$ is odd. Then there exists an integer l such that $k^2 + k + 1 = 2l + 1$. So we have

$$\begin{aligned} k^2 + k + 2 &= (k^2 + k + 1) + 1 \\ &= (2l + 1) + 1 \\ &= 2l + 2 \\ &= 2(l + 1). \end{aligned}$$

Since $l+1$ is also an integer, we conclude that k^2+k+2 is even. More importantly, we have

$$\begin{aligned} n^4+10n^2+21 &= 16(k^2+k+1)(k^2+k+2) \\ &= 16(2l+1)(2(l+1)) \\ &= 32(2l+1)(l+1) \\ &= 32(2l^2+3l+1). \end{aligned}$$

Since $2l^2+3l+1$ is also an integer, we conclude that n^4+10n^2+21 is a multiple of 32.

- Case 2: Suppose k^2+k+2 is odd. Then there exists an integer l such that $k^2+k+2=2l+1$. So we have

$$\begin{aligned} k^2+k+1 &= (k^2+k+2)-1 \\ &= (2l+1)-1 \\ &= 2l-2 \\ &= 2(l-1). \end{aligned}$$

Since $l-1$ is also an integer, we conclude that k^2+k+1 is even. More importantly, we have

$$\begin{aligned} n^4+10n^2+21 &= 16(k^2+k+1)(k^2+k+2) \\ &= 16(2(l-1))(2l+1) \\ &= 32(l-1)(2l+1) \\ &= 32(2l^2-l+1). \end{aligned}$$

Since $2l^2-l+1$ is also an integer, we conclude that n^4+10n^2+21 is a multiple of 32.

We remark that is impossible for both k^2+k+1 and k^2+k+2 to be either odd or even at the same time. The cases established that, if one of k^2+k+1 and k^2+k+2 is odd, then the other one is even, and vice versa. To try to insist that both k^2+k+1 and k^2+k+2 are either odd or even at the same time would contradict the preliminary assertions of Cases 1 and 2. Therefore, we have exhausted all possible cases of k^2+k+1 and k^2+k+2, and so we conclude that, if n is an odd integer, then n^4+10n^2+21 is a multiple of 32. □

(10pts) 3. Suppose $T \in \mathcal{L}(V,W)$ and $v \in V$. Suppose $v_1,\dots,v_n$ is a basis of V and $w_1,\dots,w_n$ is a basis of W. Prove that we have

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v).$$

Proof (3.65 of Axler). Since $v_1,\dots,v_n$ is a basis of V, it spans V, which means there exist $c_1,\dots,c_n \in \mathbb{F}$ such that every vector $v \in V$ can be written

$$v = c_1v_1+\cdots+c_nv_n.$$

So we have

$$\begin{aligned} Tv &= T(c_1v_1+\cdots+c_nv_n) \\ &= T(c_1v_1)+\cdots+T(c_nv_n) \\ &= c_1Tv_1+\cdots+c_nTv_n. \end{aligned}$$

By 3.66 of Axler, we have

$$\mathcal{M}(Tv) = c_1\mathcal{M}(Tv_1)+\cdots+c_n\mathcal{M}(Tv_n),$$

where, according to 3.64 of Axler, we can write

$$\mathcal{M}(Tv_1) = \mathcal{M}(T)_{\cdot,1},\dots,\mathcal{M}(Tv_n) = \mathcal{M}(T)_{\cdot,n}.$$

Also, by 3.62 of Axler, we have

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

which, according to 3.52 of Axler, allows us to write

$$\mathcal{M}(T)\mathcal{M}(v) = c_1\mathcal{M}(T)_{\cdot,1}+\cdots+c_n\mathcal{M}(T)_{\cdot,n}.$$

Therefore, putting all the results together, we obtain

$$\begin{aligned} \mathcal{M}(Tv) &= \mathcal{M}(c_1Tv_1+\cdots+c_nTv_n) \\ &= \mathcal{M}(c_1Tv_1)+\cdots+\mathcal{M}(c_nTv_n) \\ &= c_1\mathcal{M}(Tv_1)+\cdots+c_n\mathcal{M}(Tv_n) \\ &= c_1\mathcal{M}(T)_{\cdot,1}+\cdots+c_n\mathcal{M}(T)_{\cdot,n} \\ &= \mathcal{M}(T)\mathcal{M}(v), \end{aligned}$$

as desired. □

(10pts) 4. Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Show that V and W are isomorphic if and only if $\dim V = \dim W$.

Proof (3.59 of Axler). Forward direction: If V and W are isomorphic, then $\dim V = \dim W$. Since V and W are isomorphic, there exists an isomorphism $T \in \mathcal{L}(V, W)$; in other words, there exists an invertible linear map. Since T is invertible, it is injective and surjective, according to by 3.56 of Axler. In other words, by 3.16 of Axler we have $\operatorname{null} T = \{0\}$, and we have $\operatorname{range} T = W$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim\{0\} + \dim W \\ &= 0 + \dim W \\ &= \dim W, \end{aligned}$$

as desired.

Backward direction: If $\dim V = \dim W$, then V and W are isomorphic. Let $n = \dim V = \dim W$. Then we can let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_n$ be a basis of W. According to 3.5 of Axler (or Question 5 of Quiz 2), the map $T : V \to W$ by

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

indeed defines a function and is linear and unique. Next, we will prove that T is surjective. Since $w_1, \ldots, w_n$ is a basis of W, it spans W. So every vector in W can be written uniquely in the form $c_1 w_1 + \cdots + w_n$. Since T maps $c_1 v_1 + \cdots + c_n v_n$ to precisely $c_1 w_1 + \cdots + w_n$, it follows that T is surjective. Finally, we will prove that T is injective. Suppose we have $c_1 v_1 + \cdots + c_n v_n \in \operatorname{null} T$, which means $T(c_1 v_1 + \cdots + c_n v_n) = 0$. Then we have

$$\begin{aligned} 0 &= T(c_1 v_1 + \cdots + c_n v_n) \\ &= c_1 w_1 + \cdots + c_n w_n. \end{aligned}$$

Since $w_1, \ldots, w_n$ is linearly independent, all the scalars are zero; that is, we have

$$c_1 = 0, \ldots, c_n = 0.$$

So we get $c_1 v_1 + \cdots + c_n v_n = 0$, which means we have $\operatorname{null} T \subset \{0\}$. But 3.14 of Axler says that $\operatorname{null} T$ is a subspace in V, which means in particular that $\operatorname{null} T$ contains the additive identity, or $\{0\} \subset \operatorname{null} T$. Therefore, we have the set equality $\operatorname{null} T = \{0\}$. Finally, by 3.16 of Axler, T is injective. So we have established that T is both injective and surjective, which means, according to 3.56 of Axler, T is an isomorphism. □

(10pts) 5. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. We will prove that the following are equivalent:

(a) T is invertible;
(b) T is injective;
(c) T is surjective.

Proof (3.69 of Axler). One way to prove that statements (a), (b), (c) are equivalent statements is to prove that (a) implies (b), (b) implies (c), and (c) implies (a).

First, we will prove that (a) implies (b). Suppose (a) holds, so that T is invertible. We also have from the premises $T \in \mathcal{L}(V)$, or equivalently, that $T : V \to V$ is linear. So T is both invertible and linear, which means, according to 3.56 of Axler, T is an isomorphism, which is (b).

Next, we will prove that (b) implies (c). Suppose (b) holds, so that T is injective. Then by 3.16 of Axler we have $\operatorname{null} T = \{0\}$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &= \dim V - \dim\{0\} \\ &= \dim V - 0 \\ &= \dim V, \end{aligned}$$

which implies by 3.59 of Axler (or Question 4 of this quiz) that $\operatorname{range} T$ is isomorphic to V. But 3.19 of Axler states that $\operatorname{range} T$ is a subspace of V. Therefore, we have in fact the set equality $\operatorname{range} T = V$, and so T is surjective, which is (c).

Finally, we will prove that (c) implies (a). Suppose (c) holds, so that T is surjective. Then we have $\operatorname{range} T = V$. By the Fundamental Theorem of Linear Maps (3.22 of Axler), we have

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &= \dim V - \dim V \\ &= 0 \\ &= \dim\{0\}, \end{aligned}$$

which implies by 3.59 of Axler (or Question 4 of this quiz) that $\operatorname{null} T$ is isomorphic to $\{0\}$. But all vector spaces that are isomorphic to $\{0\}$ must be trivial themselves; in other words, we must have $\operatorname{null} T = \{0\}$. By 3.16 of Axler, T is injective. Since T is both injective and surjective, it follows that T is invertible, which is (a). □