MATH 131: Linear Algebra I
University of California, Riverside
Quiz 4 Solutions
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(10pts) 1. Use mathematical induction to prove the statement

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

for all positive integers n.

Proof. Base step: We will prove that the statement is true for $n = 1$. At $n = 1$, the statement is

$$1^2 = \frac{(1)(2(1)-1)(2(1)+1)}{3},$$

which is true because both sides of this statement equal 1.

Induction step: We will assume that the statement is true for $n = k$ and prove the statement for $n = k + 1$. The statement at $n = k$ is

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3},$$

and the statement at $n = k + 1$ is

$$1^2 + 3^2 + 5^2 + \cdots + (2(k+1)-1)^2 = \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}.$$

Indeed, using the statement for $n = k$, we have

$$\begin{aligned}
1^2 + 3^2 + 5^2 + \cdots + (2(k+1)-1)^2 &= 1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 \\
&= (1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2) + (2k+1)^2 \\
&= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \\
&= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3} \\
&= \frac{(2k+1)(k(2k-1) + 3(2k+1))}{3} \\
&= \frac{(2k+1)(2k^2+5k+3)}{3} \\
&= \frac{(2k+1)((2k+3)(k+1))}{3} \\
&= \frac{(k+1)(2k+1)(2k+3)}{3} \\
&= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3},
\end{aligned}$$

which means the statement for $n = k + 1$ holds, completing our proof by induction. □

(10pts) 2. Use mathematical induction to prove the statement

$$1^3 + 3^3 + 5^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

for all positive integers n. Use this statement to prove that every perfect cube n^3 can be written as a difference of two squares.

Proof. Base step: We will prove that the statement is true for $n = 1$. At $n = 1$, the statement is

$$1^3 = \left(\frac{(1)((1)+1)}{2}\right)^2,$$

which is true because both sides of this statement equal 1.

Induction step: We will assume that the statement is true for $n = k$ and prove the statement for $n = k + 1$. The statement at $n = k$ is

$$1^3 + 3^3 + 5^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2,$$

and the statement at $n = k + 1$ is

$$1^3 + 3^3 + 5^3 + \cdots + (k+1)^3 = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2.$$

Indeed, using the statement for $n = k$, we have

$$\begin{aligned}
1^3 + 3^3 + 5^3 + \cdots + (k+1)^3 &= (1^3 + 3^3 + 5^3 + \cdots + k^3) + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2,
\end{aligned}$$

which means the statement for $n = k + 1$ holds, completing our proof by induction. Next, we will show that n^3 can be written as a difference of two squares. Using the statement

$$1 + 3 + 5 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

we have

$$\begin{aligned}
n^3 &= (1 + 3 + 5 + \cdots + (n-1)^3 + n^3) - (1 + 3 + 5 + \cdots + (n-1)^3) \\
&= (1 + 3 + 5 + \cdots + n^3) - (1 + 3 + 5 + \cdots + (n-1)^3) \\
&= \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{(n-1)((n-1)+1)}{2}\right)^2 \\
&= \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{(n-1)n}{2}\right)^2.
\end{aligned}$$

Note that the last expression is indeed the difference of two squares. □

(10pts) 3. Use mathematical induction to prove the statement

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

for all real numbers r satisfying $r \neq 1$ and for all positive integers n.

Proof. Base step: We will prove that the statement is true for $n = 1$. At $n = 1$, the statement is

$$1 + r = \frac{r^{(1)+1} - 1}{r - 1},$$

which is true because we have

$$\begin{aligned}
\frac{r^{(1)+1} - 1}{r - 1} &= \frac{r^2 - 1}{r - 1} \\
&= \frac{(r+1)(r-1)}{r - 1} \\
&= r + 1 \\
&= 1 + r,
\end{aligned}$$

meaning the right-hand side of the above statement equals the left-hand side of the above statement.

Induction step: We will assume that the statement is true for $n = k$ and prove the statement for $n = k+1$. The statement at $n = k$ is

$$1 + r + r^2 + r^3 + \cdots + r^k = \frac{r^{k+1} - 1}{r - 1},$$

and the statement at $n = k+1$ is

$$1 + r + r^2 + r^3 + \cdots + r^{k+1} = \frac{r^{(k+1)+1} - 1}{r - 1}.$$

Indeed, using the statement for $n = k$, we have

$$\begin{aligned} 1 + r + r^2 + r^3 + \cdots + r^{k+1} &= (1 + r + r^2 + r^3 + \cdots + r^k) + r^{k+1} \\ &= \frac{r^{k+1} - 1}{r - 1} + r^{k+1} \\ &= \frac{r^{k+1} - 1 + (r-1)r^{k+1}}{r - 1} \\ &= \frac{r^{k+1} - 1 + (r^{(k+1)+1} - r^{k+1})}{r - 1} \\ &= \frac{r^{(k+1)+1} - 1}{r - 1}, \end{aligned}$$

which means the statement for $n = k+1$ holds, completing our proof by induction. □

(25pts) 4. Let V be a finite-dimensional vector space, and suppose that U is a subspace of V.

(10pts) a. Prove that the following are equivalent:

(a) $v - w \in U$;

(b) $v + U = w + U$;

(c) $(v + U) \cap (w + U) \neq \varnothing$.

Proof (3.85 of Axler). One way to prove that statements (a), (b), (c) are equivalent statements is to prove that (a) implies (b), (b) implies (c), and (c) implies (a).

First, we will prove that (a) implies (b). Suppose (a) holds, so that we have $v - w \in U$. Let $u \in U$ be arbitrary. Since U is a subspace of V, in particular it is closed under addition. This means that, since we have $v - w \in U$ from assuming (a) and we assumed $u \in U$, it follows that we must have $(v - w) + u, -(v - w) + u \in U$. So, for all $u \in U$, we have

$$\begin{aligned} v + u &= w + v - w + u \\ &= w + ((v - w) + u) \\ &\in w + U, \end{aligned}$$

which means we arrive at $v + U \subset w + U$. Similarly, for all $u \in U$, we have

$$\begin{aligned} w + u &= v + w - v + u \\ &= v + (-(v - w) + u) \\ &\in v + U, \end{aligned}$$

which means we arrive at $w + U \subset v + U$. Therefore, we have the set equality $v + U = w + U$, which is (b).

Next, we will prove that (b) implies (c). Suppose (b) holds, so that we have $v + U = w + U$. Then there exists $u \in U$ that satisfies

$$\begin{aligned} v + u &\in v + U \\ &= w + U, \end{aligned}$$

which implies $v + u \in (v + U) \cap (w + U)$. In other words, $(v + U) \cap (w + U)$ contains an element and is therefore nonempty: $(v + U) \cap (w + U) \neq \varnothing$, which is (c).

Finally, we will prove that (c) implies (a). Suppose (c) holds, so that we have $(v + U) \cap (w + U) \neq \varnothing$. Then there exist $u_1, u_2 \in U$ that satisfies

$$\begin{aligned} v + u_1 &= w + u_2 \\ &\in (v + U) \cap (w + U). \end{aligned}$$

Since U is a subspace of V, in particular it is closed under addition and scalar multiplication, which means we have $u_1 - u_2 \in U$. Therefore, we have

$$\begin{aligned} v - w &= u_2 - u_1 \\ &= -(u_1 - u_2) \\ &\in U, \end{aligned}$$

which is (a). □

(10pts) 5. Suppose U is a subspace of V. Consider on the quotient space V/U the operations of addition

$$(v + U) + (w + U) = (v + w) + U$$

and scalar multiplication

$$\lambda(v + U) = (\lambda v) + U$$

for all $v, w \in V$ and $\lambda \in \mathbb{F}$. Prove that V/U is a vector space.

Proof (3.87 of Axler). Let $v, w \in V$ be arbitrary; this means we will argue for all $v, w \in V$. First, we need to show that the operations of addition and scalar multiplication make sense on V/U. Suppose $\hat{v}, \hat{w} \in V$ satisfy

$$v + U = \hat{v} + U$$

and

$$w + U = \hat{w} + U.$$

First, we will show that addition makes sense on V/U. Since U is a subspace of V, in particular it is closed under addition, which means we have

$$\begin{aligned}(v + w) - (\hat{v} + \hat{w}) &= v + w - \hat{v} - \hat{w} \\ &= v + w + (-\hat{v}) + (-w) \\ &\in U.\end{aligned}$$

By 3.85—(a) implies (b)—of Axler, we have

$$(v + w) + U = (\hat{v} + \hat{w}) + U,$$

which confirms that addition makes sense on V/U. Now let $\lambda \in \mathbb{F}$ be arbitrary, and suppose once again that $\hat{v} \in V$ satisfies

$$v + U = \hat{v} + U.$$

Again, by 3.85 of Axler, we have $v - \hat{v} \in U$. Since U is a subspace of V, in particular it is closed under scalar multiplication, which means we have $\lambda(u - \hat{u}) \in U$, or equivalently,

$$\begin{aligned}\lambda v - \lambda\hat{v} &= \lambda(v - \hat{v}) \\ &\in U.\end{aligned}$$

By 3.85—(a) implies (b)—of Axler, we have

$$\lambda v + U = \lambda\hat{v} + U,$$

which confirms that scalar multiplication makes sense on V/U. So we have established that the operations of addition and scalar multiplication make sense on V/U. Next, we will prove that V/U is a vector space with respect to these operations.

- Commutativity: For all $v, w \in V$, we have

$$\begin{aligned}(v + U) + (w + U) &= (v + w) + U \\ &= (w + v) + U \\ &= (w + U) + (v + U).\end{aligned}$$

- Associativity: For all $v, w, x \in V$, we have

$$\begin{aligned}((v + U) + (w + U)) + (x + U) &= ((v + w) + U) + (x + U) \\ &= ((v + w) + x) + U \\ &= (v + (w + x)) + U \\ &= (v + U) + ((w + x) + U) \\ &= (v + U) + ((w + U) + (x + U)).\end{aligned}$$

- Additive identity: For all $v \in V$, we have

$$\begin{aligned}(v + U) + (0 + U) &= (v + 0) + U \\ &= v + U.\end{aligned}$$

- Additive inverse: For all $v \in V$, we have $-v \in V$ which satisfies

$$\begin{aligned}(v + U) + ((-v) + U) &= (v + (-v)) + U \\ &= 0 + U.\end{aligned}$$

- Multiplicative identity: For all $v \in V$, we have

$$\begin{aligned} 1(v + U) &= (1v) + U \\ &= v + U. \end{aligned}$$

- Distributive properties: For all $a, b \in \mathbb{F}$ and for all $v, w \in V$, we have

$$\begin{aligned} a((v + U) + (w + U)) &= a((v + w) + U) \\ &= (a(v + w)) + U \\ &= (av + aw) + U \\ &= ((av) + U) + ((aw) + U) \\ &= a(V + U) + a(w + U) \end{aligned}$$

  and

$$\begin{aligned} (a + b)(v + U) &= ((a + b)v) + U \\ &= (av + bv) + U \\ &= ((av) + U) + ((bv) + U) \\ &= a(v + U) + b(v + U). \end{aligned}$$

Since we satisfied all the properties of a vector space, we conclude that V/U is a vector space. $\square$