Math 132: Linear algebra II
Spring 2019
Midterm practice
06/13/2019 7–10pm
Time Limit: 180 Minutes

Name (Print): ____________________

Discussion TA: ____________________

Discussion time: ____________________

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1. (10 points) Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^4)$ on a complex vector space such that $T^4 = T^3$ but $T^2 \neq T$.

Solution: Let $T \in \mathbb{C}^4$ be defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$A^9 = A^8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \neq A.$$

2. (10 points) Let $V = \mathbb{C}^4$ be the vector space of 4-dimensional column vectors over the field $\mathbb{C}$. The space is equipped with **the usual dot product** in $\mathbb{C}^4$. That is

$$\left\langle \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} \right\rangle = a\overline{e} + b\overline{f} + c\overline{g} + d\overline{h}.$$

Please give an example of an operator $T \in \mathcal{L}(\mathbb{C}^4)$ such that T is normal but not self-adjoint.

Solution: Let $\mathcal{E}$ be an orthonormal basis. T is defined by $[T]_{\mathcal{E}\leftarrow\mathcal{E}} = A = \begin{bmatrix} i & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$.

It is not self-adjoint since $A^H = \begin{bmatrix} -i & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \neq A$. However $AA^H = A^H A$.

3. (10 points) Please write down ten examples of 3×3 matrix A such that $(A - 3I)^2 = 0$ but $A - 3I \neq 0$. Your answers should be either (choose ONE)

- an algorithm which is able to construct five examples
- ten explicit matrices

If you choose to give the algorithm, please describe it in details.

Solution: The Jordan form should be $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. To construct more examples, we just need to choose ten (or more) invertible matrices P and compute $P^{-1}AP$, and pick ten different matrices from the results.

4. (10 points) Let V be an inner product space. Let $S, T \in \mathcal{L}(V)$ be self-adjoint. Show that ST is self-adjoint if and only if $ST = TS$.

Solution:

($\Rightarrow$) Since S, T, ST are self-adjoint, then $S^* = S$, $T^* = T$ and $(ST)^* = ST$. Then $ST = (ST)^* = T^*S^* = TS$.

($\Leftarrow$) Since $S^* = S$, $T^* = T$ and $ST = TS$, we have $(ST)^* = T^*S^* = TS = ST$. So ST is self-adjoint.

5. (10 points) Suppose V is a complex inner product space with $V \neq \{0\}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Solution: $\overline{\lambda} \neq \lambda$ for most $\lambda \in \mathbb{C}$. Therefore if T is self-adjoint, then $(\lambda T)^* = \overline{\lambda}T^* = \overline{\lambda}T \neq \lambda T$ in general. So self-adjoint operators don't form a subspace of $\mathcal{L}(V)$.

6. (10 points) Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^4$. Let $\mathcal{E} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis of V and $\mathcal{F} = \left\{ f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, f_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, f_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis of W. Let $T \in \mathcal{L}(V, W)$ be defined by the matrix

$$A = [T]_{\mathcal{F} \leftarrow \mathcal{E}} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Please find a basis of V and a basis of W such that the matrix of T under the new basis is as simple as possible. That is, the entries should contain as many zeros as possible, and if the entry is not zero, try your best to make it to be 1, if able.

Solution: The linear map tells us that

$$\begin{aligned} \mathcal{T}(e_1) &= f_1 + 2f_2 + 3f_3 + f_4, \\ \mathcal{T}(e_2) &= 3f_1 + f_2, \\ \mathcal{T}(e_3) &= 2f_1 + 4f_2 + f_3 + 2f_4. \end{aligned}$$

It is easy to check that $\{f_1 + 2f_2 + 3f_3 + f_4, 3f_1 + f_2, 2f_1 + 4f_2 + f_3 + 2f_4\}$ is linearly independent. Then let

$$w_1 = f_1 + 2f_2 + 3f_3 + f_4, \quad w_2 = 3f_1 + f_2, \quad w_3 = 2f_1 + 4f_2 + f_3 + 2f_4.$$

$\{w_1, w_2, w_3\}$ forms a linearly independent set in W, which is then a basis. Now extend it to be a basis $\mathcal{C}$ in W by adding $w_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Now we have

$$\begin{aligned} \mathcal{T}(e_1) &= w_1, \\ \mathcal{T}(e_2) &= w_2, \\ \mathcal{T}(e_3) &= w_3. \end{aligned}$$

Then the matrix of the linear map is

$$[\mathcal{T}]_{\mathcal{C} \leftarrow \mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix is simple enough therefore we don't need to change the basis of V.

Then the basis we use is $\mathcal{E}$ for V and $\mathcal{C}$ for W.

7. (10 points) Let $V = \mathbb{C}^3$ be the vector space of 3-dimensional column vectors over the field $\mathbb{C}$. Let

$$\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

be the standard basis of V. Let $T \in \mathcal{L}(V)$ be defined by the matrix A as $\left[T\right]_{\mathcal{S}\leftarrow\mathcal{S}} = A$:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 2 \\ -1 & -2 & -4 \end{bmatrix}.$$

Please find a basis $\mathcal{B} = \{v_1, v_2, v_3\}$ such that $\left[T\right]_{\mathcal{B}\leftarrow\mathcal{B}}$ is an upper-triangular matrix. Please write down the resulted upper-triangular matrix **using coordinates method**.

Solution:

1. First compute the characteristic polynomial $\det(A - xI) = -(x+2)^3$. Therefore the only eigenvalue is -2.

2. Find all eigenvectors. Solve $(A - (-2)I)X = 0$.

   The solution space is $\texttt{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right)$.

3. Pick $v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. These are the first two basis vectors.

4. Consider $V/\texttt{Span}(v_1, v_2)$. $\left\{ \left[e_1\right] \right\}$ is a basis. Then it has to be an eigenvector. The class $\left[e_1\right] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \texttt{Span}(v_1, v_2)$. The third basis vector should be any non-zero representative in it. Pick $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

5. The basis $\mathcal{B} = \left\{ v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis that $\left[T\right]_{\mathcal{B}\leftarrow\mathcal{B}}$ is an upper-triangular matrix.

6. $T(v_3) = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 2 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = a \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then we have

$$\begin{bmatrix} -2 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.$$

Solve for a, b, c, we have $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$. Therefore $\left[T(v_3)\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$. Since v_1, v_2 are two eigenvectors with respect to eigenvalue -2, $T(v_1) = -2v_1$ and $T(v_2) = -2v_2$. Then $\left[T(v_1)\right]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ and $\left[T(v_2)\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$. Then

$$\left[T\right]_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -2 \end{bmatrix}.$$

8. (10 points) Let $V = \mathbb{C}^3$ be the vector space of 3-dimensional column vectors over the field $\mathbb{C}$. Let

$$\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

be the standard basis of V. Let $T \in \mathcal{L}(V)$ be defined by the matrix A as $\big[T\big]_{\mathcal{S}\leftarrow\mathcal{S}} = A$:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 2 \\ -1 & -2 & -4 \end{bmatrix}.$$

Please find a Jordan basis $\mathcal{C} = \{w_1, w_2, w_3\}$ of this operator. Please write down the Jordan form in your basis **using change-of-basis method**.

Solution:

1. From the previous problem, $\mathcal{B} = \left\{ v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis that $\big[T\big]_{\mathcal{B}\leftarrow\mathcal{B}}$ is an upper-triangular matrix.

2. $A - (-2)I = \begin{bmatrix} -2-(-2) & 0 & 0 \\ 1 & 0-(-2) & 2 \\ -1 & -2 & -4-(-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ -1 & -2 & -2 \end{bmatrix}.$

3. $w_3 = v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$

4. $(A-(-2)I)w_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \neq 0.$

   Then $w_2 = (A-(-2)I)w_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$

5. $(A-(-2)I)w_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0.$ Then $w_1 = v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$

6. Then $\mathcal{C} = \left\{ w_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a Jordan basis.

7. The change-of-basis matrix $P_{\mathcal{S}\leftarrow C} = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$. Then

$$[T]_{\mathcal{C}\leftarrow\mathcal{C}} = P_{\mathcal{S}\leftarrow\mathcal{C}}^{-1}[T]_{\mathcal{C}\leftarrow\mathcal{C}}P_{\mathcal{S}\leftarrow\mathcal{C}} = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 2 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

This is the Jordan canonical form of T.

9. (10 points) Let $V = \mathbb{C}^3$ be the vector space of 3-dimensional column vectors over the field $\mathbb{C}$. Let

$$\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

be the standard basis of V. Let $T \in \mathcal{L}(V)$ be defined by the matrix A as $\big[T\big]_{\mathcal{S}\leftarrow\mathcal{S}} = A$:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 2 \\ -1 & -2 & -4 \end{bmatrix}.$$

Please write down the Jordan form of T **using polynomials method**. You are NOT allowed to compute the basis.

Solution:

1. First compute the characteristic polynomial $\det(A - xI) = -(x+2)^3$.
2. From the characteristic polynomial, the only possible minimal polynomial is $(x+2)$ or $(x+2)^2$. Since $A + 2I = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 2 \\ -1 & -2 & -4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ -1 & -2 & -2 \end{bmatrix} \neq 0$, the minimal polynomial has to be $(x+2)^2$.
3. From these two polynomials, we know that the Jordan form has the following properties:
   - The only eigenvalue is -2.
   - The biggest size of Jordan blocks is 2.

   Therefore there are two Jordan block with eigenvalue -2, with size 1 and 2. Then the Jordan form of T is $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

10. (10 points) Let $V = \mathbb{C}^3$ be the vector space of 3-dimensional column vectors over the field $\mathbb{C}$. The space is equipped with **the usual dot product** in $\mathbb{C}^3$. That is

$$\left\langle \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix} \right\rangle = a\overline{d} + b\overline{e} + c\overline{f}.$$

Let

$$\mathcal{F} = \left\{ f_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, f_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, f_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

be a basis of V. Let $T \in \mathcal{L}(V)$ be defined by the matrix A as $\left[T\right]_{\mathcal{F}\leftarrow\mathcal{F}} = A$:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & -1 \\ 0 & 0 & -3 \end{bmatrix}.$$

Please find an ORTHORNORMAL basis $\mathcal{B} = \{v_1, v_2, v_3\}$ such that $\left[T\right]_{\mathcal{B}\leftarrow\mathcal{B}}$ is an upper-triangular matrix. You **don't need** to write the resulted upper-triangular matrix you get.

Solution: Since the matrix of T is already upper-triangular, all we need to do is to apply the Gram-Schmidt procedure to change $\mathcal{E}$ into an orthonormal basis.

1. $v_1 = \frac{f_1}{\|f_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$

2. $w_2 = f_2 - \langle f_2, v_1 \rangle v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}.$

   Then $v_2 = \frac{w_2}{\|w_2\|} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}.$

3.

$$\begin{aligned} w_3 &= f_3 - \langle f_3, v_1 \rangle v_1 - \langle f_3, v_2 \rangle v_2 \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left\langle \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} - \left\langle \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}. \end{aligned}$$

Then $v_3 = \frac{w_3}{\|w_3\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$.

To sum up, $\mathcal{B} = \{v_1, v_2, v_3\} = \left\{ v_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, v_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}, v_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$ is an orthonormal basis such that $\left[T\right]_{\mathcal{B} \leftarrow \mathcal{B}}$ is an upper-triangular matrix.