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MATH 132 Homework 1

3.1. Find the coordinate of the vector $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ with respect to the basis

$$\mathcal{U} = \left\{ u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Proof. For all $i = 1, 2, 3$, there exist coefficients $x_1, x_2, x_3 \in \mathbb{R}$ such that we can write any $x \in \mathbb{R}^3$ as

$$\begin{aligned} x &= x_1 u_1 + x_2 u_2 + x_3 u_2 \\ &= x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} [x]_{\mathcal{U}}. \end{aligned}$$

With the vector $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, the corresponding coordinate vector $[x]_{\mathcal{U}}$ is

$$\begin{aligned} [x]_{\mathcal{U}} &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}. \end{aligned}$$

Alternatively, if one wishes to avoid taking an inverse of a 3×3 matrix, one can rewrite the matrix equation as an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right],$$

which, upon performing a few row operations, is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right],$$

whose right-most column is $[x]_{\mathcal{U}}$. □

3.2. Let P_3 be the vector space of all polynomials of variable t with degree no higher than 3. Find the matrix representation for taking derivative $D : P_3 \to P_3$ with respect to the basis

$$\mathcal{F} = \{f_1 = t^3, \quad f_2 = t^3 + t^2, \quad f_3 = t^3 + t^2 + t, \quad f_4 = t^3 + t^2 + t + 1\}.$$

Proof. Since D is a linear map that denotes taking the derivative in t of f_1, f_2, f_3, f_4, we have

$$D(f_1) = 3t^2, \quad D(f_2) = 3t^2 + 2t, \quad D(f_3) = D(f_4) = 3t^2 + 2t + 1.$$

Let $a_1, a_2, a_3, a_4 \in \mathbb{R}$ and write

$$\begin{aligned} a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 &= a_1 t^3 + a_2(t^3 + t^2) + a_3(t^3 + t^2 + t) + a_4(t^3 + t^2 + t + 1) \\ &= (a_1 + a_2 + a_3 + a_4)t^3 + (a_2 + a_3 + a_4)t^2 + (a_3 + a_4)t + (a_4)1. \end{aligned}$$

First, for $D(f_1) = 3t^2$, we have

$$\begin{aligned} 3t^2 &= a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 \\ &= (a_1 + a_2 + a_3 + a_4)t^3 + (a_2 + a_3 + a_4)t^2 + (a_3 + a_4)t + (a_4)1. \end{aligned}$$

We can equate the coefficients to rewrite this as a system of equations:

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 0 \\ a_2 + a_3 + a_4 &= 3 \\ a_3 + a_4 &= 0 \\ a_4 &= 0. \end{aligned}$$

The augmented matrix form of our system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right],$$

which, upon performing row operations, is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

So our desired coefficients are $a_1 = -3, a_2 = 3, a_3 = 0, a_4 = 0$; in other words,

$$\left[D(f_1)\right]_{\mathcal{F}} = \begin{bmatrix} -3 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

Next, for $D(f_2) = 3t^2 + 2t$, we have

$$\begin{aligned} 3t^2 + 2t &= a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 \\ &= (a_1 + a_2 + a_3 + a_4)t^3 + (a_2 + a_3 + a_4)t^2 + (a_3 + a_4)t + (a_4)1. \end{aligned}$$

We can equate the coefficients to rewrite this as a system of equations:

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 0 \\ a_2 + a_3 + a_4 &= 3 \\ a_3 + a_4 &= 2 \\ a_4 &= 0. \end{aligned}$$

The augmented matrix form of our system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right],$$

which, upon performing row operations, is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

So our desired coefficients are $a_1 = -3, a_2 = 1, a_3 = 2, a_4 = 0$; in other words,

$$\left[D(f_2)\right]_{\mathcal{F}} = \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

Finally, for $D(f_3) = D(f_4) = 3t^2 + 2t + 1$, we have

$$\begin{aligned} 3t^2 + 2t + 1 &= a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 \\ &= (a_1 + a_2 + a_3 + a_4)t^3 + (a_2 + a_3 + a_4)t^2 + (a_3 + a_4)t + (a_4)1. \end{aligned}$$

We can equate the coefficients to rewrite this as a system of equations:

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 0 \\ a_2 + a_3 + a_4 &= 3 \\ a_3 + a_4 &= 2 \\ a_4 &= 0. \end{aligned}$$

The augmented matrix form of our system is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right],$$

which, upon performing row operations, is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

So our desired coefficients are $a_1 = -3, a_2 = 1, a_3 = 1, a_4 = 1$; in other words,

$$\left[D(f_3)\right]_{\mathcal{F}} = \left[D(f_4)\right]_{\mathcal{F}} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} [D]_{\mathcal{F}\leftarrow\mathcal{F}} &= \left[\left[D(f_1)\right]_{\mathcal{F}} \quad \left[D(f_2)\right]_{\mathcal{F}} \quad \left[D(f_3)\right]_{\mathcal{F}} \quad \left[D(f_4)\right]_{\mathcal{F}}\right] \\ &= \begin{bmatrix} -3 & -3 & -3 & -3 \\ 3 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

is the matrix representation of $D : P^3 \to P^3$ with respect to $\mathcal{F}$. □

3.3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

for all $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Consider the basis

$$\mathcal{B} = \left\{ x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(1) Compute the matrix representation of A with respect to $\{x_1, x_2\}$.

Proof. We have

$$\begin{aligned} T(x_1) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} T(x_2) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{aligned}$$

Let $a_1, a_2 \in \mathbb{R}$ and write

$$\begin{aligned} a_1 x_1 + a_2 x_2 &= a_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \end{aligned}$$

First, for $T(x_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

The augmented matrix form of our system is

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right],$$

which, upon performing row operations, is row equivalent to

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right].$$

So our desired coefficients are $a_1 = 0$, $a_2 = 1$; in other words,

$$\left[T(x_1)\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Next, for $T(x_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

The augmented matrix form of our system is

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right],$$

which, upon performing row operations, is row equivalent to

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right].$$

So our desired coefficients are $a_1 = -1$, $a_2 = 0$; in other words,

$$\left[T(x_2)\right]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} [T]_{\mathcal{B}\leftarrow\mathcal{B}} &= \begin{bmatrix} \left[T(x_1)\right]_{\mathcal{B}} & \left[T(x_2)\right]_{\mathcal{B}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

is the matrix representation of T with respect to $\{x_1, x_2\}$. □

(2) Use the matrix $[T]_{\mathcal{B}\leftarrow\mathcal{B}}$ from part (1) to compute $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$.

Proof. In order to compute $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ using $[T]_{\mathcal{B}\leftarrow\mathcal{B}}$, we have to first compute $\left[T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\right]_{\mathcal{B}}$ using the matrix $[T]_{\mathcal{B}\leftarrow\mathcal{B}}$. From the equation

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \left[\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

we obtain the coordinate vector

$$\begin{aligned} \left[\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right]_{\mathcal{B}} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}. \end{aligned}$$

Therefore, according to Theorem 3.5.1 of Xinli Xiao's lecture notes, we have

$$\begin{aligned} \left[T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\right]_{\mathcal{B}} &= [T]_{\mathcal{B}\leftarrow\mathcal{B}} \left[\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right]_{\mathcal{B}} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix}, \end{aligned}$$

from which we finally obtain

$$\begin{aligned} T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) &= -\frac{3}{2}x_1 - \frac{1}{2}x_2 \\ &= -\frac{3}{2}\begin{bmatrix}1\\-1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} \\ &= \begin{bmatrix}-2\\1\end{bmatrix}, \end{aligned}$$

as desired. This can be verified quickly using the definition of the linear map T:

$$\begin{aligned} T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) &= \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} \\ &= \begin{bmatrix}-2\\1\end{bmatrix}, \end{aligned}$$

which is the same answer. □