Exercise 3.1. Find the coordinate of the vector $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ with respect to the basis

$$\mathcal{U} = \left\{ u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Exercise 3.2. Let P_3 be the vector space of all polynomials of variable t with degree no higher than 3. Find the matrix representation for taking derivative $D : P_3 \to P_3$ with respect to the basis

$$\mathcal{F} = \{ f_1 = t^3, \quad f_2 = t^3 + t^2, \quad f_3 = t^3 + t^2 + t, \quad f_4 = t^3 + t^2 + t + 1 \}.$$

Exercise 3.3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map defined by

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{for } \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2.$$

Consider the basis

$$\mathcal{B} = \left\{ x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(1) Compute the matrix representation of T with respect to $\mathcal{B}$.

(2) Use the above matrix (computed from Part (1)) to compute $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$.

The homework is due on Apr. 5.

3. Solutions to Exercise 1

Exercise 3.1. Find the coordinate of the vector $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ with respect to the basis

$$\mathcal{U} = \left\{ u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution 3.1. To find the $\mathcal{U}$-coordinate, we need to find the coefficients x_1, x_2, x_3 such that

$$x = x_1 u_1 + x_2 u_2 + x_3 u_3.$$

Then we have a linear system:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

That is

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} x \end{bmatrix}_{\mathcal{U}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Exercise 3.2. Let P_3 be the vector space of all polynomials of variable t with degree no higher than 3. Find the matrix representation for taking derivative $D : P_3 \to P_3$ with respect to the basis

$$\mathcal{F} = \{f_1 = t^3, \quad f_2 = t^3 + t^2, \quad f_3 = t^3 + t^2 + t, \quad f_4 = t^3 + t^2 + t + 1\}.$$

Solution 3.2. Apply D to all basis vectors:

$$D(f_1) = 3t^2, \quad D(f_2) = 3t^2 + 2t, \quad D(f_3) = 3t^2 + 2t + 1, \quad D(f_4) = 3t^2 + 2t + 1.$$

We need to find the $\mathcal{F}$-coordinates of each result. Let us first compute $x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4$:

$$\begin{aligned} & x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 \\ =& x_1(t^3) + x_2(t^3 + t^2) + x_3(t^3 + t^2 + t) + x_4(t^3 + t^2 + t + 1) \\ =& (x_1 + x_2 + x_3 + x_4)t^3 + (x_2 + x_3 + x_4)t^2 + (x_3 + x_4)t + x_4. \end{aligned}$$

$D(f_1)$**:** The equation is $3t^2 = x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4$. Then we have

$$3t^2 = (x_1 + x_2 + x_3 + x_4)t^3 + (x_2 + x_3 + x_4)t^2 + (x_3 + x_4)t + x_4.$$

That is

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0, \\ x_2 + x_3 + x_4 &= 3, \\ x_3 + x_4 &= 0, \\ x_4 &= 0. \end{aligned}$$

Then we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

The solution is $x_1 = -3,\ x_2 = 3,\ x_3 = 0,\ x_4 = 0$. Therefore

$$\Big[D(f_1)\Big]_{\mathcal{F}} = \begin{bmatrix} -3 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

$D(f_2)$**:** The equation is $3t^2 + 2t = x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4$. Then we have

$$3t^2 + 2t = (x_1 + x_2 + x_3 + x_4)t^3 + (x_2 + x_3 + x_4)t^2 + (x_3 + x_4)t + x_4.$$

That is

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0, \\ x_2 + x_3 + x_4 &= 3, \\ x_3 + x_4 &= 2, \\ x_4 &= 0. \end{aligned}$$

Then we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

Then solution is $x_1 = -3,\ x_2 = 1,\ x_3 = 2,\ x_4 = 0$. Therefore

$$\Big[D(f_2)\Big]_{\mathcal{F}} = \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

$D(f_3)$**:** The equation is $3t^2 + 2t + 1 = x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4$. Then we have

$$3t^2 + 2t + 1 = (x_1 + x_2 + x_3 + x_4)t^3 + (x_2 + x_3 + x_4)t^2 + (x_3 + x_4)t + x_4.$$

That is

$$x_1 + x_2 + x_3 + x_4 = 0,$$

$$x_2 + x_3 + x_4 = 3,$$

$$x_3 + x_4 = 2,$$

$$x_4 = 1.$$

Then we have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

Then solution is $x_1 = -3$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$. Therefore

$$\Big[D(f_3)\Big]_{\mathcal{F}} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$D(f_4)$**:** The equation is $3t^2 + 2t + 1 = x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4$. This is actually the same equation as $D(f_3)$-case. Then we have

$$\Big[D(f_4)\Big]_{\mathcal{F}} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

To sum up, the matrix of D with respect to the basis $\mathcal{F}$ of both domain and codomain is

$$\Big[D\Big]_{\mathcal{F} \leftarrow \mathcal{F}} = \begin{bmatrix} -3 & -3 & -3 & -3 \\ 3 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Exercise 3.3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map defined by

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{for} \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2.$$

Consider the basis

$$\mathcal{B} = \left\{ x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(1) Compute the matrix representation of T with respect to $\mathcal{B}$.

Solution 3.3. Apply T on the basis vectors of $\mathcal{B}$:

$$T(x_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T(x_2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Now we want to find the $\mathcal{B}$-coordinates of these two vectors.

$T(x_1)$**:** The equation is $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = a_1 x_1 + a_2 x_2 = a_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. The solution is $a_1 = 0$, $a_2 = 1$. Then

$$\left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$T(x_2)$**:** The equation is $\begin{bmatrix} -1 \\ 1 \end{bmatrix} = a_1 x_1 + a_2 x_2 = a_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. The solution is $a_1 = -1$, $a_2 = 0$. Then

$$\left[\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

To sum up, the matrix of T with respect to $\mathcal{B}$ is

$$\left[T\right]_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(2) Use the above matrix (computed from Part (1)) to compute $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$.

Solution 3.3. We first find the $\mathcal{B}$-coordinate of $\begin{bmatrix}1\\2\end{bmatrix}$. That is, to solve the equation

$$\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}.$$

Then solution is $a_1 =$, $a_2 =$. Then

$$\left[\begin{bmatrix}1\\2\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}-1/2\\3/2\end{bmatrix}.$$

Then

$$\left[T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right]_{\mathcal{B}} = \left[T\right]_{\mathcal{B}\leftarrow\mathcal{B}}\left[\begin{bmatrix}1\\2\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}-1/2\\3/2\end{bmatrix} = \begin{bmatrix}-3/2\\-1/2\end{bmatrix}.$$

Then

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = -\frac{3}{2}x_1 - \frac{1}{2}x_2 = -\frac{3}{2}\begin{bmatrix}1\\-1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}.$$