MATH 132 Homework 2

4.1. Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^4$, and let $\mathcal{T} : V \to W$ be a linear transformation defined by the matrix

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Please find a good basis for both V and W to make the matrix of $\mathcal{T}$ as simple as possible. Please also write down the change-of-basis matrix and the matrix of $\mathcal{T}$ under the new basis.

Proof. Consider the standard basis of V

$$\mathcal{S}_V = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and the standard basis of W

$$\mathcal{S}_W = \left\{ f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, f_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, f_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Then

$$\mathcal{T}(e_1) = Ae_1 = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = -f_1 + f_2,$$

$$\mathcal{T}(e_2) = Ae_2 = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 2f_1 + f_2 - f_3,$$

$$\mathcal{T}(e_3) = Ae_3 = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} = 3f_1 - 2f_3 + f_4.$$

Define

$$\begin{aligned} w_1 &= -f_1 + f_2, \\ w_2 &= 2f_1 + f_2 - f_3, \\ w_3 &= 3f_1 + 2f_3 + f_4, \end{aligned}$$

so that

$$\begin{aligned} \mathcal{T}(e_1) &= -f_1 + f_2 = w_1, \\ \mathcal{T}(e_2) &= 2f_1 + f_2 - f_3 = w_2, \\ \mathcal{T}(e_3) &= 3f_1 - 2f_3 + f_4 = w_3. \end{aligned}$$

We claim that the set $\{w_1, w_2, w_3\} \subset W$ is linearly independent if $\{f_1, f_2, f_3\} \subset W$ is linearly independent. Indeed, if $a_1, a_2, a_3 \in \mathbb{R}$ satisfy

$$a_1w_1 + a_2w_2 + a_3w_3 = 0,$$

then we obtain an equivalent equation

$$(-a_1 + 2a_2 + 3a_3)f_1 + a_2f_2 + (a_1 + 2a_3)f_3 + a_3f_4 = 0.$$

Since $\{f_1, f_2, f_3\} \subset W$ is linearly independent, we must have the system of equations

$$\begin{aligned} -a_1 + 2a_2 + 3a_3 &= 0 \\ a_2 &= 0 \\ a_1 + 2a_3 &= 0 \\ a_3 &= 0. \end{aligned}$$

The only solution to this system of equations is $a_1 = a_2 = a_3 = 0$, which proves our claim. Now, we consider another vector

$$w_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then, since the matrix $B = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}$ is invertible (because $\det(B) = 1 \neq 0$), we conclude that $C_W = \{w_1, w_2, w_3, w_4\} \subset W$ is a linearly independent set that spans W. Hence, $C_W = \{w_1, w_2, w_3, w_4\} \subset W$ is a basis. Now, we have

$$\begin{aligned} \left[\mathcal{T}\right]_{C_W \leftarrow C_V} &= \begin{bmatrix} [\mathcal{T}(e_1)]_{C_W} & [\mathcal{T}(e_2)]_{C_W} & [\mathcal{T}(e_3)]_{C_W} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

And the change of basis matrix is

$$\begin{aligned} P_{\mathcal{S}_W \leftarrow C_W} &= \begin{bmatrix} [w_1]_{\mathcal{S}_W} & [w_2]_{\mathcal{S}_W} & [w_3]_{\mathcal{S}_W} & [w_4]_{\mathcal{S}_W} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \end{aligned}$$

as desired. □

4.2. Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$, and let $\mathcal{T} : V \longrightarrow W$ be a linear transformation defined by the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -3 & -1 \end{bmatrix}.$$

Please find a good basis for both V and W to make the matrix of $\mathcal{T}$ as simple as possible. Please also write down the change-of-basis matrix and the matrix of $\mathcal{T}$ under the new basis.

Proof. Consider the standard basis of V

$$\mathcal{S}_V = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and the standard basis of W

$$\mathcal{S}_W = \left\{ f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Then

$$\mathcal{T}(e_1) = Ae_1 = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = f_1 - f_2,$$

$$\mathcal{T}(e_2) = Ae_2 = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 3f_1 - 3f_2,$$

$$\mathcal{T}(e_3) = Ae_3 = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2f_1 - f_2.$$

Define

$$\begin{aligned} w_1 &= f_1 - f_2, \\ w_2 &= 2f_1 - f_2, \end{aligned}$$

so that

$$\begin{aligned} \mathcal{T}(e_1) &= f_1 - f_2 = w_1, \\ \mathcal{T}(e_2) &= 3f_1 - 3f_2 = 3(f_1 - f_2) = 3w_1, \\ \mathcal{T}(e_3) &= 2f_1 - f_2 = w_2. \end{aligned}$$

We claim that the set $\{w_1, w_2\} \subset W$ is linearly independent if $\{f_1, f_2\} \subset W$ is linearly independent. Indeed, if $a_1, a_2 \in \mathbb{R}$ satisfy

$$a_1 w_1 + a_2 w_2 = 0,$$

then we obtain an equivalent equation

$$(a_2 + 2a_2) f_1 + (-a_1 - a_2) f_2 = 0.$$

Since $\{f_1, f_2, f_3\} \subset W$ is linearly independent, we must have the system of equations

$$\begin{aligned} a_1 + 2a_2 &= 0, \\ -a_1 - a_2 &= 0. \end{aligned}$$

The only solution to this system of equations is $a_1 = a_2 = 0$, which proves our claim. Furthermore, $\{w_1, w_2\} \subset W$ also forms a basis because any vector in W can be expressed as a linear combination of w_1, w_2, which is another way of saying that $\{w_1, w_2\}$ spans W. Now, if we want to simplify the matrix A, also define

$$\begin{aligned} v_1 &= e_1, \\ v_2 &= e_3, \\ v_3 &= e_2 - 3e_1, \end{aligned}$$

so that

$$\begin{aligned} \mathcal{T}(v_1) &= \mathcal{T}(e_1) = w_1, \\ \mathcal{T}(v_2) &= \mathcal{T}(e_3) = w_2, \\ \mathcal{T}(v_3) &= \mathcal{T}(e_2 - 3e_1) = \mathcal{T}(e_2) - 3\mathcal{T}(e_1) = 3w_1 - 3w_1 = 0. \end{aligned}$$

(The motivation for defining v_1, v_2, v_3 above was that the image under $\mathcal{T}$ of v_3 is zero, which makes sense since $\mathcal{T}$ sends $(v_1, v_2, v_3) \in V$ to $(w_1, w_2) \in W$.) We claim that the set $\mathcal{C}_V = \{v_1, v_2, v_3\} \subset V$ is linearly independent if $\{e_1, e_2, e_3\} \subset V$ is linearly independent. Indeed, if $c_1, c_2, c_3 \in \mathbb{R}$ satisfy

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0,$$

then we obtain an equivalent equation

$$(c_1 - 3c_3) e_1 + c_3 e_2 + c_2 e_3 = 0.$$

Since $\{f_1, f_2, f_3\} \subset W$ is linearly independent, we must have the system of equations

$$\begin{aligned} c_1 - 3c_3 &= 0, \\ c_3 &= 0, \\ c_2 &= 0. \end{aligned}$$

The only solution to this system of equations is $c_1 = c_2 = c_3 = 0$, which proves our claim. Furthermore, $\mathcal{C}_V = \{v_1, v_2, v_3\} \subset V$ also forms a basis because any vector in V can be expressed as a linear combination of v_1, v_2, v_3, which is another way of saying that $\{v_1, v_2, v_3\}$ spans V. Now, we find

$$\begin{aligned} \left[\mathcal{T}\right]_{\mathcal{C}_W \leftarrow \mathcal{C}_V} &= \begin{bmatrix} \left[\mathcal{T}(e_1)\right]_{\mathcal{C}_W} & \left[\mathcal{T}(e_2)\right]_{\mathcal{C}_W} & \left[\mathcal{T}(e_1)\right]_{\mathcal{C}_W} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

The change of basis matrix on V is

$$\begin{aligned} P_{\mathcal{S}_V \leftarrow \mathcal{C}_V} &= \begin{bmatrix} \left[v_1\right]_{\mathcal{S}_V} & \left[v_2\right]_{\mathcal{S}_V} & \left[v_3\right]_{\mathcal{S}_V} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \end{aligned}$$

and the change of basis matrix on W is

$$\begin{aligned} P_{\mathcal{S}_W \leftarrow \mathcal{C}_W} &= \begin{bmatrix} \left[w_1\right]_{\mathcal{S}_W} & \left[w_2\right]_{\mathcal{S}_W} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \end{aligned}$$

as desired. $\square$

4.3. Let $V = W = \mathbb{R}^2$, and let $\mathcal{T} : V \to W$ be a linear transformation defined by the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Please find a good basis for both V and W to make the matrix of $\mathcal{T}$ as simple as possible. Please also write down the change-of-basis matrix and the matrix of $\mathcal{T}$ under the new basis.

Proof. Consider the standard basis

$$\mathcal{S}_V = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Then

$$\begin{aligned} \mathcal{T}(e_1) = Ae_1 &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e_1 - e_2, \\ \mathcal{T}(e_2) = Ae_2 &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -e_1 + e_2. \end{aligned}$$

To simplify the matrix A, consider also another set

$$\mathcal{B} = \{v_1 = ae_1 + ce_2, v_2 = be_1 + de_2\},$$

which is also linearly independent and hence a basis. Since $\mathcal{B}$ is a basis, the matrix

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible, which means that the determinant of this matrix is nonzero, i.e. $ad - bc \neq 0$. So we can system-solve for e_1, e_2 in terms of v_1, v_2; the explicit expressions are

$$\begin{aligned} e_1 &= \frac{d}{ad-bc} v_1 - \frac{c}{ad-bc} v_2, \\ e_2 &= -\frac{b}{ad-bc} v_1 + \frac{a}{ad-bc} v_2. \end{aligned}$$

So we have

$$\begin{aligned} \mathcal{T}(v_1) &= \mathcal{T}(ae_1 + ce_2) \\ &= a\mathcal{T}(e_1) + c\mathcal{T}(e_2) \\ &= a(e_1 - e_2) + c(-e_1 + e_2) \\ &= a\left(\left(\frac{d}{ad-bc}v_1 - \frac{c}{ad-bc}v_2\right) - \left(-\frac{b}{ad-bc}v_1 + \frac{a}{ad-bc}v_2\right)\right) \\ &\quad + c\left(-\left(\frac{d}{ad-bc}v_1 - \frac{c}{ad-bc}v_2\right) + \left(-\frac{b}{ad-bc}v_1 + \frac{a}{ad-bc}v_2\right)\right) \\ &= \frac{ad + ab - cd - cb}{ad-bc} v_1 + \frac{-ac - a^2 - c^2 + ca}{ad-bc} v_2 \\ &= \frac{(a-c)(b+d)}{ad-bc} v_1 - \frac{(a+c)(a-c)}{ad-bc} v_2 \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}(v_2) &= \mathcal{T}(be_1 + de_2) \\ &= b\mathcal{T}(e_1) + d\mathcal{T}(e_2) \\ &= b(e_1 - e_2) + d(-e_1 + e_2) \\ &= b\left(\left(\frac{d}{ad-bc}v_1 - \frac{c}{ad-bc}v_2\right) - \left(-\frac{b}{ad-bc}v_1 + \frac{a}{ad-bc}v_2\right)\right) \\ &\quad + d\left(-\left(\frac{d}{ad-bc}v_1 - \frac{c}{ad-bc}v_2\right) + \left(-\frac{b}{ad-bc}v_1 + \frac{a}{ad-bc}v_2\right)\right) \\ &= \frac{bd + b^2 - d^2 - db}{ad-bc} v_1 + \frac{-bc - ba - dc + da}{ad-bc} v_2 \\ &= \frac{(b+d)(b-d)}{ad-bc} v_1 - \frac{(a+c)(b-d)}{ad-bc} v_2. \end{aligned}$$

Therefore,

$$\begin{aligned}
\left[\mathcal{T}\right]_{\mathcal{B}\leftarrow\mathcal{B}} &= \left[\left[\mathcal{T}(v_1)\right]_{\mathcal{B}} \quad \left[\mathcal{T}(v_2)\right]_{\mathcal{B}}\right] \\
&= \left[\tfrac{(a-c)(b+d)}{ad-bc}\left[v_1\right]_{\mathcal{B}} - \tfrac{(a+c)(a-c)}{ad-bc}\left[v_2\right]_{\mathcal{B}} \quad \tfrac{(b+d)(b-d)}{ad-bc}\left[v_1\right]_{\mathcal{B}} - \tfrac{(a+c)(b-d)}{ad-bc}\left[v_2\right]_{\mathcal{B}}\right] \\
&= \frac{1}{ad-bc}\left[(a-c)(b+d)\left[v_1\right]_{\mathcal{B}} - (a+c)(a-c)\left[v_2\right]_{\mathcal{B}} \quad (b+d)(b-d)\left[v_1\right]_{\mathcal{B}} - (a+c)(b-d)\left[v_2\right]_{\mathcal{B}}\right] \\
&= \frac{1}{ad-bc}\left[(a-c)(b+d)\begin{bmatrix}1\\0\end{bmatrix} - (a+c)(a-c)\begin{bmatrix}0\\1\end{bmatrix} \quad (b+d)(b-d)\begin{bmatrix}1\\0\end{bmatrix} - (a+c)(b-d)\begin{bmatrix}0\\1\end{bmatrix}\right] \\
&= \frac{1}{ad-bc}\begin{bmatrix}(a-c)(b+d) & (b+d)(b-d)\\ -(a+c)(a-c) & -(a+c)(b-d)\end{bmatrix}.
\end{aligned}$$

For the matrix $\left[\mathcal{T}\right]_{\mathcal{B}\leftarrow\mathcal{B}}$ to be simple enough, we can impose that $\left[\mathcal{T}\right]_{\mathcal{B}\leftarrow\mathcal{B}}$ needs to be a diagonal matrix, which would require entries not on the main diagonal to be zero; in other words,

$$\begin{aligned}
(b+d)(b-d) &= 0, \\
-(a+c)(a-c) &= 0.
\end{aligned}$$

This system of equations yields the solutions $a = \pm c$ and $b = \pm d$. For simplicity, we can choose $a = b = d = 1$ and $c = -1$. Then

$$\begin{aligned}
v_1 &= ae_1 + ce_2 = e_1 - e_2, \\
v_2 &= be_1 + de_2 = e_1 + e_2.
\end{aligned}$$

So

$$\begin{aligned}
\left[\mathcal{T}\right]_{\mathcal{C}_W\leftarrow\mathcal{C}_V} &= \left[\left[\mathcal{T}(e_1)\right]_{\mathcal{C}_W} \quad \left[\mathcal{T}(e_2)\right]_{\mathcal{C}_W}\right] \\
&= \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}.
\end{aligned}$$

The change of basis matrix on V is

$$\begin{aligned}
P_{\mathcal{S}_V\leftarrow\mathcal{C}_V} &= \left[\left[\mathcal{T}(V_1)\right]_{\mathcal{S}_V} \quad \left[\mathcal{T}(v_2)\right]_{\mathcal{S}_V}\right] \\
&= \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix},
\end{aligned}$$

as desired. □