4. Solutions to Exercise 4

Exercise 4.1. Let $\mathcal{T} : V \to W$ be a linear transformation from $V = \mathbb{R}^3$ to $W = \mathbb{R}^4$ defined by the matrix $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. Please find a good basis for both V and W to make the matrix of $\mathcal{T}$ as simple as possible. Please also write down the change-of-basis matrices and the matrix of $\mathcal{T}$ under the new basis.

Solution 4.1. We start from the standard basis:

$$\mathcal{S}_V = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{S}_W = \left\{ f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, f_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, f_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Then the linear map tells us that

$$\mathcal{T}(e_1) = -f_1 + f_3,$$

$$\mathcal{T}(e_2) = 2f_1 + f_2 - f_3,$$

$$\mathcal{T}(e_3) = 3f_1 - 2f_3 + f_4.$$

It is easy to check that $\{-f_1 + f_3, 2f_1 + f_2 - f_3, 3f_1 - 2f_3 + f_4\}$ is linearly independent (**but you should show this in your solution**). Then let

$$w_1 = -f_1 + f_3, \quad w_2 = 2f_1 + f_2 - f_3, \quad w_3 = 3f_1 - 2f_3 + f_4.$$

$\{w_1, w_2, w_3\}$ forms a linearly independent set in W. Now extend it to be a basis $\mathcal{C}_W$ in W by adding $w_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (**and you need to show why you can do this in your solution**). Now we have

$$\mathcal{T}(e_1) = w_1,$$

$$\mathcal{T}(e_2) = w_2,$$

$$\mathcal{T}(e_3) = w_3.$$

Then the matrix of the linear map is

$$\left[\mathcal{T}\right]_{\mathcal{C}_W \leftarrow \mathcal{S}_V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix is simple enough therefore we don't need to change the basis of V. The change-of-basis matrix on W is then

$$P_{\mathcal{S}_W \leftarrow \mathcal{C}_W} = \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Exercise 4.2. Let $\mathcal{T} : V \to W$ be a linear transformation from $V = \mathbb{R}^3$ to $W = \mathbb{R}^2$, defined by the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -3 & -1 \end{bmatrix}$. Please find a good basis for both V and W to make the matrix of $\mathcal{T}$ as simple as possible. Please also write down the change-of-basis matrices and the matrix of $\mathcal{T}$ under the new basis.

Solution 4.2. We start from the standard basis:

$$\mathcal{S}_V = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{S}_W = \left\{ f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Then the linear map tells us that

$$\mathcal{T}(e_1) = f_1 - f_2, \quad \mathcal{T}(e_2) \ = 3f_1 - 3f_2, \quad \mathcal{T}(e_3) = 2f_1 - f_2.$$

It is easy to check that $\{f_1 - f_2, 2f_1 - f_2\}$ is linearly independent (**but you should show this in your solution**). Then let

$$w_1 = f_1 - f_2, \quad w_2 = 2f_1 - f_2.$$

$\{w_1, w_2\}$ forms a basis $\mathcal{C}_W$ of W (**and you need to show this in your solution**). Now we have

$$\mathcal{T}(e_1) = w_1, \quad \mathcal{T}(e_2) \; = 3w_1, \quad \mathcal{T}(e_3) = w_2.$$

Then we need to change the basis of V to simplify the matrix further. Since $\mathcal{T}(e_2) = 3w_1 = 3\mathcal{T}(e_1)$, we have $\mathcal{T}(e_2 - 3e_1) = 0$. Then we choose

$$v_1 = e_1, \quad v_2 = e_3, \quad v_3 = e_2 - 3e_1.$$

It is easy to check that $\mathcal{C}_V = \{v_1, v_2, v_3\}$ forms a basis of V **(but you need to show it in your solution)**. Then we have

$$\mathcal{T}(v_1) = w_1, \quad \mathcal{T}(v_2) = w_2, \quad \mathcal{T}(v_3) = 0.$$

Therefore the matrix of $\mathcal{T}$ under the bases $\mathcal{C}_V$ and $\mathcal{C}_W$ is

$$\left[\mathcal{T}\right]_{\mathcal{C}_W \leftarrow \mathcal{C}_V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The change-of-basis matrix on V is then

$$P_{\mathcal{S}_V \leftarrow \mathcal{C}_V} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The change-of-basis matrix on W is then

$$P_{\mathcal{S}_W \leftarrow \mathcal{C}_W} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}.$$

Exercise 4.3. Let $\mathcal{T}: V \to W$ be a linear transformation from $V = \mathbb{R}^2$ to itself, defined by the matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Please find a good basis for V to make the matrix of $\mathcal{T}$ as simple as possible. Please also write down the change-of-basis matrices and the matrix of $\mathcal{T}$ under the new basis.

Solution 4.3. We start from the standard basis:

$$\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Then the linear map tells us that

$$\mathcal{T}(e_1) = e_1 - e_2, \quad \mathcal{T}(e_2) \quad = -e_1 + e_2.$$

Now we want to choose another basis to simplify the matrix. So we set another basis

$$\mathcal{B} = \{v_1 = ae_1 + ce_2, \quad v_2 = be_1 + de_2\}.$$

Since $\mathcal{B}$ is a basis, it is easy to check that $ad - bc \neq 0$. Then we have

$$e_1 = \frac{d}{ad-bc}v_1 + \frac{-c}{ad-bc}v_2, \quad e_2 = \frac{-b}{ad-bc}v_1 + \frac{a}{ad-bc}v_2.$$

Then

$$\begin{aligned}
\mathcal{T}(v_1) &= \mathcal{T}(ae_1 + ce_2) = a\mathcal{T}(e_1) + c\mathcal{T}(e_2) = a(e_1 - e_2) + c(-e_1 + e_2) = (a-c)e_1 + (-a+c)e_2 \\
&= \frac{(a-c)d + (-a+c)(-b)}{ad-bc}v_1 + \frac{(a-c)(-c) + (-a+c)a}{da-bc}v_2 \\
&= \frac{(a-c)(d+b)}{ad-bc}v_1 + \frac{(a-c)(-c-a)}{ad-bc}v_2 = \frac{1}{ad-bc}((a-c)(b+d)v_1 - (a-c)(a+c)v_2), \\
\mathcal{T}(v_2) &= \mathcal{T}(be_1 + de_2) = b\mathcal{T}(e_1) + d\mathcal{T}(e_2) = b(e_1 - e_2) + d(-e_1 + e_2) = (b-d)e_1 + (-b+d)e_2 \\
&= \frac{(b-d)d + (-b+d)(-b)}{ad-bc}v_1 + \frac{(b-d)(-c) + (-b+d)a}{da-bc}v_2 \\
&= \frac{(b-d)(d+b)}{ad-bc}v_1 + \frac{(b-d)(-c-a)}{ad-bc}v_2 = \frac{1}{ad-bc}((b-d)(b+d)v_1 - (b-d)(a+c)v_2).
\end{aligned}$$

So

$$\left[\mathcal{T}\right]_{\mathcal{B}\leftarrow\mathcal{B}} = \frac{1}{ad-bc}\begin{bmatrix} (a-c)(b+d) & (b-d)(b+d) \\ -(a-c)(a+c) & -(b-d)(a+c) \end{bmatrix}.$$

If we want this matrix to be simple, we hope it to be diagonal. That is,

$$(b-d)(b+d) = 0, \quad -(a-c)(a+c) = 0.$$

So $b = \pm d$ and $a = \pm c$. There are actually 4 solutions. We only need one. Let us pick $b = d = 1$ and $a = -c = 1$. (Think: why not choose $b = d$ and $a = c$?) Then $v_1 = e_1 - e_2$ and $v_2 = e_1 + e_2$. We have

$$\left[\mathcal{T}\right]_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_{\mathcal{S}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Remark 4.0.1. You can try, but you will see that it is impossible to make 2 to be 1 by changing basis.