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MATH 132 Homework 3

5.1. Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Proof. Suppose to the contrary that some operator has $n > \dim V$ distinct eigenvalues. Then there exist n linearly independent eigenvectors that correspond with the n distinct eigenvalues, which contradicts the fact that V has at most $\dim V$ linearly independent vectors. Hence, V has at most $\dim V$ distinct eigenvalues. □

5.2. (1) Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\text{nul}(S)$ is invariant under T.

Proof. Let $v \in \text{nul}(S)$ be arbitrary; i.e. $Sv = 0$. Then, since T is a linear map, we have $STv = TSv = T(0) = 0$, which means $Tv \in \text{nul}(S)$. Hence, $\text{nul}(S)$ is invariant under T. □

(2) Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\text{im}(S)$ is invariant under T.

Proof. Let $v \in \text{im}(S)$ be arbitrary; i.e. there exists $u \in V$ such that $S(u) = v$. Then we have $STu = TSu = Tv$, which means $Tv \in \text{im}(S)$. Hence, $\text{im}(S)$ is invariant under T. □

5.3. See the proof of Theorem 5.4.1. Let $v \in V$. Let $\{[v_1], \ldots, [v_k]\} \subset V/\text{span}(V)$ be a basis. Please show that $\{v, v_1, \ldots, v_k\} \subset V$ is linearly independent, using the definition of linear independence.

Proof. To show that $\{v, v_1, \ldots, v_k\}$ is linearly independent in V, we need to assume that we have $cv + c_1v_1 + \cdots + c_kv_k = 0$ for some scalars $c, c_1, \ldots, c_k \in \mathbb{C}$ and then show that $c = c_1 = \cdots = c_k = 0$. To this end, we have from our assumption $c_1v_1 + \cdots + c_kv_k = -cv \in \text{span}(v)$, which means we have

$$\begin{aligned} 0 &= [c_1v_1 + \cdots + c_kv_k] \\ &= c_1[v_1] + \cdots + c_k[v_k]. \end{aligned}$$

Since the hypothesis states that $\{[v_1], \ldots, [v_k]\} \subset V/\text{span}(V)$ is a basis, it follows that the equation $c_1[v_1] + \cdots + c_k[v_k] = 0$ implies $c_1 = \cdots = c_k = 0$. Returning to our equation $cv + c_1v_1 + \cdots + c_kv_k = 0$, we conclude $c = 0$. Hence, we established $c = c_1 = \cdots = c_k = 0$, which means $\{v, v_1, \ldots, v_k\} \subset V$ is linearly independent. □

5.4. Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T has an invariant subspace of dimension k for each $k = 1, \ldots, \dim V$.

Proof. Theorem 5.4.1 of the Notes (also see 5.27 of Axler) asserts that there exists some basis $\mathcal{B} = \{v_1, \ldots, v_{\dim V}\}$ of V such that we can write $\left[T\right]_{\mathcal{B}\leftarrow\mathcal{B}}$ as an upper triangular matrix. Furthermore, by Remark 5.4.10 of the Notes (also see 5.26 of Axler), we have

$$\begin{aligned} Tv_1 &\in \text{span}(v_1) \\ Tv_2 &\in \text{span}(v_1, v_2) \\ Tv_3 &\in \text{span}(v_1, v_2, v_3) \\ &\vdots \\ Tv_{\dim V} &\in \text{span}(v_1, \ldots, v_{\dim V}). \end{aligned}$$

Since $T(v_k)$ is a linear combination of $v_1, \ldots, v_k$, which is equivalent to saying $Tv_k \in \text{span}(v_1, \ldots, v_k)$, it follows that $\text{span}(v_1, \ldots, v_k) \subset V$ is an invariant subspace of dimension k. □

5.5. Suppose W is a complex vector space and $T \in \mathcal{L}(W)$ has no eigenvalues. Prove that every subspace of W invariant under T is either $\{0\}$ or infinite-dimensional.

Proof. Let $U \subset W$ be a subspace that is invariant under T, i.e. for all $v \in U$ we have $T(y) \in Y$. Suppose to the contrary that $U \subset W$ is a non-zero and finite-dimensional subspace. Then there exist a non-zero $w \in U$ and $\lambda \in \mathbb{C}$ such that $Tw = \lambda w$; i.e. there exists an eigenvector $w \in U$. But since we have $U \subset W$, it follows that w is an eigenvector in W as well, which implies that there exists a corresponding eigenvalue λ of T. But this contradicts our assumption that T has no eigenvalues. Hence, U is either $\{0\}$ or infinite-dimensional. □

5.6. Let $T \in \mathcal{L}(\mathbb{C}^3)$ be defined by the matrix

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & 0 \\ 1 & 6 & 2 \end{bmatrix}.$$

Find a basis of $\mathbb{C}^3$ that expresses T as an upper-triangular matrix.

Proof. First, we will consider the standard basis

$$\mathcal{S} := \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Then the change of basis matrix is

$$\begin{aligned} \left[T\right]_{\mathcal{S}\leftarrow\mathcal{S}} &= \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & 0 \\ 1 & 6 & 2 \end{bmatrix}. \end{aligned}$$

To find the eigenvalues of T, we have

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} 2-\lambda & -2 & 0 \\ 0 & 3-\lambda & 0 \\ 1 & 6 & 2-\lambda \end{bmatrix} \\ &= (2-\lambda)^2(3-\lambda), \end{aligned}$$

from which we obtain our eigenvalues $\lambda = 2$ and $\lambda = 3$. For $\lambda = 2$, we have

$$\begin{aligned} 0 &= (A - 2I)x \\ &= \begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

So our solution is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} \in \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and so we can choose $v_1 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to be an eigenvector for $\lambda = 2$. Next, we consider the quotient space $V/\text{span}(v_1)$. We note that

$$\left\{ [e_1] := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \text{span}(v_1), [e_2] := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \text{span}(v_1) \right\}$$

is a basis of $V/\text{span}(v_1)$. We also have $V/\text{span}(v_1) \cong \mathbb{C}^2$ because the map from $V/\text{span}(v_1)$ to $\mathbb{C}^2$ defined by $[e_1] \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [e_2] \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an isomorphism; this observation will be useful a bit later. Now we consider the linear map $\overline{T} : V/\text{span}(v_1) \to V/\text{span}(v_1)$ between the quotient spaces. We have

$$\begin{aligned} \overline{T}([e_1]) &= [T(e_1)] \\ &= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \text{span}(v_1) \\ &= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &= 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &= 2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right) \\ &= 2[e_1] \end{aligned}$$

and

$$\begin{aligned}
\overline{T}([e_2]) &= [T(e_2)] \\
&= \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} + \operatorname{span}(v_1) \\
&= \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} + \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\
&= -2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\
&= -2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right) + 3 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right) \\
&= 2[e_1] + 3[e_2].
\end{aligned}$$

Using the isomorphism that we introduced just earlier, the matrix representation of $\overline{T}$ is

$$\begin{aligned}
B &= \begin{bmatrix} \overline{T}([e_1]) & \overline{T}([e_2]) \end{bmatrix} \\
&= \begin{bmatrix} 2[e_1] & -2[e_1] + 3[e_2] \end{bmatrix} \\
&= \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix}.
\end{aligned}$$

To find the eigenvalues of $\overline{T}$, we have

$$\begin{aligned}
0 &= \det(B - \lambda I) \\
&= \det \begin{bmatrix} 2-\lambda & -2 \\ 0 & 3-\lambda \end{bmatrix} \\
&= (2-\lambda)(3-\lambda),
\end{aligned}$$

from which we get our eigenvalues $\lambda = 2$ and $\lambda = 3$. For $\lambda = 2$, we have

$$\begin{aligned}
0 &= (B - 2I)x \\
&= \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\end{aligned}$$

So our solution is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \in \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Due to our isomorphism from $V/\operatorname{span}(v_1)$ to $\mathbb{R}^2$, we have that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \operatorname{span}(v_1)$ is an eigenvector of $\overline{T}$, with $v_2 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ being a representative of this class. Now, we consider the quotient space $V/\operatorname{span}(v_1, v_2)$. We note that

$$\left\{ [e_2] := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \operatorname{span}(v_1, v_2) \right\}$$

is a basis of $V/\operatorname{span}(v_1, v_2)$. We also have $V/\operatorname{span}(v_1, v_2) \cong \mathbb{C}$ because the map from $V/\operatorname{span}(v_1, v_2)$ to $\mathbb{C}$ defined by $[e_2] \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$ is an isomorphism. Now we consider the linear map $\overline{T} : V/\operatorname{span}(v_1, v_2) \to V/\operatorname{span}(v_1, v_2)$ between the quotient spaces. We have

$$\begin{aligned}
\overline{T}([e_2]) &= [T(e_2)] \\
&= \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} + \operatorname{span}(v_1, v_2) \\
&= \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} + \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \\
&= 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \\
&= 3 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \right) \\
&= 3[e_2],
\end{aligned}$$

which signifies that $\lambda = 3$ is an eigenvalue of $\overline{T}$ and $[e_2] := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \operatorname{span}(v_1, v_2)$ is an eigenvector of $\overline{T}$ corresponding to $\lambda = 3$, with $v_3 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ being a representative of this class. So the basis of V consisting of these representatives is

$$\mathcal{B} := \left\{ v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

So the change of basis matrix is

$$P_{\mathcal{S} \leftarrow \mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Hence, the upper triangular matrix of T is

$$\begin{aligned}
\left[T\right]_{\mathcal{B} \leftarrow \mathcal{B}} &= P_{\mathcal{B} \leftarrow S} \left[T\right]_{\mathcal{S} \leftarrow \mathcal{S}} \left[P\right]_{\mathcal{S} \leftarrow \mathcal{B}} \\
&= P^{-1}_{\mathcal{S} \leftarrow B} \left[T\right]_{\mathcal{S} \leftarrow \mathcal{S}} \left[P\right]_{\mathcal{S} \leftarrow \mathcal{B}} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & 0 \\ 1 & 6 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & 0 \\ 1 & 6 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix},
\end{aligned}$$

as desired. ◻