5. Solutions to Exercise 5

Exercise 5.1. Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Solution 5.1. Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Let $v_1, \ldots, v_m$ be corresponding eigenvectors. Then $\{v_1, \ldots, v_m\}$ is linearly independent. Thus $m \leq \dim V$.

Exercise 5.2.

(1) Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\texttt{Nul}(S)$ is invariant under T.

Solution 5.2. For any $v \in \texttt{Nul}(S)$, $S(v) = 0$. Since $ST = TS$, $S(T(v)) = T(S(v)) = T(0) = 0$. Then $T(v) \in \texttt{Nul}(S)$. Then $\texttt{Nul}(S)$ is invariant under T.

(2) Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\texttt{im}(S)$ is invariant under T.

Solution 5.2. For any $v \in \texttt{im}(S)$, there exists $w \in V$ such that $S(w) = v$. Since $ST = TS$, $T(v) = T(S(w)) = S(T(w))$. Then $T(v) \in \texttt{im}(S)$. Then $\texttt{im}(S)$ is invariant under T.

Exercise 5.3. See the proof of Theorem 5.4.1. Let $v \in V$. Let $\{[v_1], \ldots, [v_k]\}$ be a basis $V/\texttt{Span}(v)$. Please show that $\{v, v_1, \ldots, v_k\}$ is linearly independent using the definition of linearly independence.

Solution 5.3. Set up the equation in V:

$$cv + c_1v_1 + \ldots + c_kv_k = 0. \tag{**}$$

Then $c_1v_1 + \ldots + c_kv_k = -cv \in \texttt{Span}(v)$. Then $[c_1v_1 + \ldots + c_kv_k] = 0 \in V/\texttt{Span}(v)$. So

$$c_1[v_1] + \ldots + c_k[v_k] = [c_1v_1 + \ldots + c_kv_k] = 0.$$

Since $\{[v_1], \ldots, [v_k]\}$ is a basis of $V/\texttt{Span}(v)$, $c_1 = \ldots = c_k = 0$. Then in Equation (**), we have $cv = 0$. Therefore $c = 0$. Then $\{v, v_1, \ldots, v_k\}$ is linearly independent. $\square$

Exercise 5.4. Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T has an invariant subspace of dimension k for each $k = 1, \ldots, \dim V$.

Proof. Let $n = \dim V$. By Theorem 5.4.1, there is a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ such that $\Big[T\Big]_{\mathcal{B}\leftarrow\mathcal{B}}$ is upper-triangular. Then

$$T(v_1) \in \mathtt{Span}(v_1),$$

$$T(v_2) \in \mathtt{Span}(v_1, v_2),$$

$$\cdots\cdots$$

$$T(v_n) \in \mathtt{Span}(v_1, \ldots, v_n).$$

Then $W_k = \mathtt{Span}(v_1, \ldots, v_k)$ is the desired T-invariant subspace of dimension k for any $k = 1, \ldots, n$. $\square$

Exercise 5.5. Suppose W is a complex vector space and $T \in \mathcal{L}(W)$ has no eigenvalues. Prove that every subspace of W invariant under T is either $\{0\}$ or infinite-dimensional.

Proof. Use contradiction. Assume that there is a finite-dimensional non-zero subspace U of W invariant under T. Then by Theorem 5.4.2, $T|_U$ has an eigenvector in U. Then there exists $w \in U$, $\lambda \in \mathbb{C}$ such that $w \neq 0$ and $T(w) = \lambda w$. Since $w \in U \subset W$, w is an eigenvector of T in W. This is a contradiction. So the assumption is wrong. Then every subspace of W invariant under T is either $\{0\}$ or infinite-dimensional. $\square$

Exercise 5.6. Let $T \in \mathcal{L}(\mathbb{C}^3)$ which is defined by the matrix

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & 0 \\ 1 & 6 & 2 \end{bmatrix}.$$

Find a basis $\mathbb{C}^3$ to write T as an upper-triangular matrix.

Solution 5.6. Let $V = \mathbb{C}^3$. Let $\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis.

Then

$$\Big[T\Big]_{\mathcal{S}\leftarrow\mathcal{S}} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 3 & 0 \\ 1 & 6 & 2 \end{bmatrix}.$$

(1) Find an eigenvector of T on V: $\det(A - \lambda I) = 0$. Then $(2-\lambda)^2(3-\lambda) = 0$. So 2 and 3 are eigenvalues. We use 2 as our current eigenvalue here. Solve the equation $(A - 2I)X = 0$ for $X \in \mathbb{C}^3$.

$$\begin{bmatrix} 2-2 & -2 & 0 \\ 0 & 3-2 & 0 \\ 1 & 6 & 2-2 \end{bmatrix} X = 0.$$

Solutions are $X \in \texttt{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$. We take $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as our first eigenvector.

(2) Consider the quotient space $V/\texttt{Span}(v_1)$. Consider the basis $\left\{\left[e_1\right], \left[e_2\right]\right\}$ where

$$\left[e_1\right] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \texttt{Span}(v_1), \quad \left[e_2\right] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \texttt{Span}(v_1).$$

$V/\texttt{Span}(v_1) \simeq \mathbb{C}^2$ by $\left[e_1\right] \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\left[e_2\right] \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(3) Since

$$\overline{T}\left(\left[e_1\right]\right) = \left[T(e_1)\right] = \left[\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right] = 2\left[e_1\right],$$

$$\overline{T}\left(\left[e_2\right]\right) = \left[T(e_2)\right] = \left[\begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}\right] = -2\left[e_1\right] + 3\left[e_2\right].$$

Therefore under the basis $\left\{\left[e_1\right], \left[e_2\right]\right\}$, the matrix of $\overline{T}$ is $B = \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix}$.

(4) Find an eigenvector of $\overline{T}$ on $V/\texttt{Span}(v_1)$: $\det(B - \lambda I) = 0$. Then $(2-\lambda)(3-\lambda) = 0$. So 2 and 3 are eigenvalues. We use 2 as our current eigenvalue here. Solve the equation

$(B-2I)X = 0$ for $X \in \mathbb{C}^2 \simeq V/\,\mathtt{Span}(v_1)$.

$$\begin{bmatrix} 2-2 & -2 \\ 0 & 3-2 \end{bmatrix} X = 0.$$

Solutions are $X \in \mathtt{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. We take $\overline{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. By the isomorphism between $\mathbb{C}^2$ and $V/\,\mathtt{Span}(v_1)$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is corresponding to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mathtt{Span}(v_1)$. We take a representative from the class to be v_2. For example we can pick $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(5) Consider the quotient space $V/\,\mathtt{Span}(v_1, v_2)$. Consider the basis $\left\{\left[e_2\right]\right\}$ where

$$\left[e_2\right] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mathtt{Span}(v_1, v_2).$$

$V/\,\mathtt{Span}(v_1, v_2) \simeq \mathbb{C}$ by $\left[e_2\right] \mapsto 1$.

(6) Since

$$\overline{T}\left(\left[e_2\right]\right) = \left[T(e_2)\right] = \left[\begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}\right] = 3\left[e_2\right].$$

Then $\begin{bmatrix} e_2 \end{bmatrix}$ is an eigenvector of $\overline{T}$ in $V/\texttt{Span}(v_1, v_2)$. The vector is corresponding to the class $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \texttt{Span}(v_1, v_2)$. We take a representative from the class to be v_3. For example we can pick $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(7) We now have a basis $\mathcal{B} = \left\{ v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. The change-of-basis $P_{\mathcal{S} \leftarrow \mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Then the matrix of T is

$$\begin{bmatrix} T \end{bmatrix}_{\mathcal{B} \leftarrow \mathcal{B}} = P_{\mathcal{S} \leftarrow \mathcal{B}}^{-1} \begin{bmatrix} T \end{bmatrix}_{\mathcal{S} \leftarrow \mathcal{S}} P_{\mathcal{S} \leftarrow \mathcal{B}} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}.$$