Ryan Ta

MATH 132 Homework 4

5.7. Check whether the following matrices are diagonalizable. Note that you do NOT need compute the diagonalized matrix or the change-of-basis matrix.

(1) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$

Proof. We can solve the equation

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 5-\lambda & 3 \\ 0 & 0 & 8-\lambda \end{bmatrix}\right) \\ &= (2-\lambda)(5-\lambda)(7-\lambda) \end{aligned}$$

to obtain our eigenvalues $\lambda = 2, 5, 8$. All our eigenvalues are distinct, which implies that there exists a basis of eigenvectors. (Recall on a side note that the converse is generally false.) We conclude using Theorem 5.5.2 of the Notes that A is diagonalizable. □

(2) $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Proof. We can solve the equation

$$\begin{aligned} 0 &= \det(B - \lambda I) \\ &= \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)(4-\lambda) - (3)(2) \\ &= \lambda^2 - 5\lambda - 2 \end{aligned}$$

to obtain our eigenvalues $\lambda = \frac{5-\sqrt{33}}{2}, \frac{5+\sqrt{33}}{2}$. All our eigenvalues are distinct, which implies that there exists a basis of eigenvectors. (Recall on a side note that the converse is generally false.) We conclude using Theorem 5.5.2 of the Notes that A is diagonalizable. □

(3) $C = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Proof. We can solve the equation

$$\begin{aligned} 0 &= \det(C - \lambda I) \\ &= \det\left(\begin{bmatrix} 2-\lambda & -1 & 0 \\ 0 & 3-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix}\right) \\ &= (2-\lambda)^2(3-\lambda) \end{aligned}$$

to obtain our eigenvalues $2, 2, 3$. Not all of our eigenvalues are distinct, which means C may or may not be diagonalizable. If C were diagonalizable, then we would be able to find a basis of eigenvectors; otherwise, if we cannot find such a basis, then C would not be diagonalizable. For $\lambda = 2$, we have

$$\begin{aligned} 0 &= (C - 2I)x \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$

So a solution can be $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$, and so we can choose $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to be an eigenvector for $\lambda = 2$. For $\lambda = 3$, we have

$$\begin{aligned} 0 &= (C - 3I)x \\ &= \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$

So a solution can be $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}$, and so we can choose $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ to be an eigenvector for $\lambda = 3$. We are able to find only two linearly independent eigenvalues, which means there does not exist a basis of eigenvectors. We conclude using Theorem 5.5.2 of the Notes that C is not diagonalizable. □

5.8. (1) Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{nul} T \oplus \operatorname{im} T$.

Proof. Since T is diagonalizable, by implication (1) $\Rightarrow$ (2) Theorem 5.5.2 of the Notes, V has a basis consisting of eigenvectors $v_1, \ldots, v_n$. So there exist eigenalues $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that $Tv_i = \lambda_i v_i$ for all $i = 1, \ldots, n$. We note an obvious fact that an eigenvalue is either zero or nonzero. Furthermore, there exists $k \in \{1, \ldots, n\}$ such that k of our n eigenvalues $\lambda_1, \ldots, \lambda_n$ is zero (this does not contradict Theorem 5.5.2 of the Notes); in this case, our remaning $n - k$ eigenvalues are nonzero. Without loss of generality, suppose $\lambda_i = 0$ for all $i = 1, \ldots, k$ and $\lambda_i \neq 0$ for all $i = k+1, \ldots, n$. If $i = 1, \ldots, k$, then $\lambda_i = 0$, which means

$$\begin{aligned} Tv_i &= \lambda_i v_i \\ &= 0v_i \\ &= 0, \end{aligned}$$

and so $v_1, \ldots, v_k \in \operatorname{nul} T$, which implies $\operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{nul} T$. If $i = k+1, \ldots, n$, then $\lambda_i \neq 0$, which means

$$\begin{aligned} T\left(\frac{1}{\lambda_i} v_i\right) &= \lambda_i \left(\frac{1}{\lambda_i} v_i\right) \\ &= v_i, \end{aligned}$$

and so $v_{k+1}, \ldots, v_n \in \operatorname{im} T$, which implies $\operatorname{span}\{v_{k+1}, \ldots, v_n\} = \operatorname{im} T$. Hence,

$$\begin{aligned} V &= \operatorname{span}\{v_1, \ldots, v_n\} \\ &= \operatorname{span}\{v_1, \ldots, v_k\} \oplus \operatorname{span}\{v_{k+1}, \ldots, v_n\} \\ &= \operatorname{nul} T \oplus \operatorname{im} T, \end{aligned}$$

as desired. □

(2) State the converse of the statement above. Prove it or give a counterexample.

Proof. Converse: If $V = \operatorname{nul} T \oplus \operatorname{im} T$, then T is diagonalizable. We will disprove the converse. For our counterexample, let $V = \mathbb{C}^3$ and represent T by the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} Te_1 &= Ae_1 \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

and so $e_1 \in \operatorname{nul} T$. Solving the equation $Av = 0$ implies $\operatorname{span}\{e_1\} = \operatorname{nul} T$. We also have

$$\begin{aligned} Te_2 &= Ae_2 \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} Te_3 &= Ae_3 \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \end{aligned}$$

and so $e_2, e_3 \in \operatorname{im} T$, which implies $\operatorname{span}\{e_2, e_3\} = \operatorname{im} T$. □

5.9. Give an example that $R, T \in \mathcal{L}(\mathbb{C}^4)$ such that R and T each have $2, 6, 7$ as eigenvalues and no other eigenvalues, and there does not exist an invertible operator $S \in \mathcal{L}(\mathbb{C}^4)$ such that $R = S^{-1}TS$.

Proof. Consider the two bases $\mathcal{E} := \{e_1, e_2, e_3, e_4\}, \mathcal{F} := \{f_1, f_2, f_3, f_4\} \subset \mathbb{C}^4$. Since $2, 6, 7$ are the only eigenvalues of S, T, we can choose a matrix representation of R with respect to $\mathcal{E}$ to be

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

and a matrix representation of T with respect to $\mathcal{F}$ to be

$$B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

Based on the matrix representations A, B, we find that R is diagonalizable but T is not diagonalizable. Assume to the contrary that there exists an invertible operator $S \in \mathcal{L}(\mathbb{C}^4)$ such that $R = S^{-1}TS$. Let $v_1 = S^{-1}(f_1)$ and $v_2 = S^{-1}(f_2)$. Since $\{f_1, f_2\} \subset \mathbb{C}^2$ is a linearly independent set and S is a linear map, it follows that $\{v_1, v_2\} \subset \mathbb{C}^2$ is also a linearly independent set. But we also have

$$\begin{aligned} Rv_1 &= S^{-1}TSv_1 \\ &= S^{-1}TS(S^{-1}f_1) \\ &= S^{-1}Tf_1 \\ &= S^{-1}(2f_1) \\ &= 2S^{-1}(f_1) \\ &= 2v_1 \end{aligned}$$

and

$$\begin{aligned} Rv_2 &= S^{-1}Tsv_2 \\ &= S^{-1}TS(S^{-1}f_2) \\ &= S^{-1}Tf_2 \\ &= S^{-1}(2f_2) \\ &= 2S^{-1}(f_2) \\ &= 2v_2, \end{aligned}$$

which means that v_1, v_2 are both eigenvectors of R corresponding to $\lambda = 2$. On the other hand, since 2 is an eigenvalue of R, we have

$$\begin{aligned} 0 &= (A - 2I)v \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} v \end{aligned}$$

which implies $v \in \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$, and so any eigenvector of R must be a scalar of multiple of $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. So the contradiction here is that an argument using the bases $\mathcal{E}, \mathcal{F}$ of $\mathbb{C}^4$ asserts that R has two linearly independent eigenvectors, but at the same time our direct computation establishes that R has only one linearly independent eigenvector. Hence, the invertible operator $S \in \mathcal{L}(\mathbb{C}^4)$ does not exist. □

5.10. Let V be finite-dimensional, and $S, T \in \mathcal{L}(V)$. Suppose T has $\dim V$ distinct eigenvalues, and S has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

Proof. Let $\lambda_1, \ldots, \lambda_{\dim V} \in \mathbb{F}$ be the $\dim V$ distinct eigenvalues of T. Then there exist $\dim V$ linear independent vectors $v_1, \ldots, v_{\dim V}$, which by definition satisfy $Tv_i = \lambda_i v_i$ for all $i = 1, \ldots, \dim V$. Since S has the same eigenvectors $v_1, \ldots, v_{\dim V}$ as T, there exist corresponding eigenvalues $\mu_1, \ldots, \mu_{\dim V} \in \mathbb{F}$ such that $Sv_i = \mu_i v_i$ for all $i = 1, \ldots, \dim V$. Now we consider an arbitrary vector $v \in V$. Since $\{v_1, \ldots, v_{\dim V}\} \subset V$ is a basis, we can write $v = c_1 v_1 + \cdots + c_{\dim V} v_{\dim V}$ for some $c_1, \ldots, c_{\dim V} \in \mathbb{F}$. So for all $v \in V$ we have

$$\begin{aligned} STv &= ST(c_1 v_1 + \cdots + c_{\dim V} v_{\dim V}) \\ &= S(c_1 T v_1 + \cdots + c_{\dim V} T v_{\dim V}) \\ &= S(c_1 \lambda_1 v_1 + \cdots + c_{\dim V} \lambda_{\dim V} v_{\dim V}) \\ &= c_1 \lambda_1 S v_1 + \cdots + c_n \lambda_{\dim V} S_{\dim V} \\ &= c_1 \lambda_1 \mu_1 v_1 + \cdots + c_{\dim V} \lambda_{\dim V} \mu_{\dim V} v_{\dim V} \end{aligned}$$

and

$$\begin{aligned} TSv &= TS(c_1 v_1 + \cdots + c_{\dim V} v_{\dim V}) \\ &= T(c_1 S v_1 + \cdots + c_{\dim V} S v_{\dim V}) \\ &= S(c_1 \mu_1 v_1 + \cdots + c_{\dim V} \mu_{\dim V} v_{\dim V}) \\ &= c_1 \mu_1 S v_1 + \cdots + c_n \mu_{\dim V} S_{\dim V} \\ &= c_1 \mu_1 \lambda_1 v_1 + \cdots + c_{\dim V} \mu_{\dim V} \lambda_{\dim V} v_{\dim V}. \end{aligned}$$

So we proved $STv = TSv$ for all $v \in V$, which is equivalent to proving $ST = TS$. □

5.11. The *Fibonacci sequence* $F_1, F_2, \ldots$ is defined by

$$F_1 = 1, F_2 = 1, F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 3,$$

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x + y \end{bmatrix}.$$

Note: For parts (2)-(4), I am presenting a solution to finding eigenvalues and eigenvectors that is different from your instructor's, without finding a matrix representation of the linear map T. This matrix-free approach is how the textbook author Sheldon Axler wanted his readers to do this exercise. However, this solution here is only for your amusement, because for your MATH 132 class your instructor would like you to follow the standard matrix approach to this problem, among other problems.

(1) Show that

$$T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

for each positive integer n.

Proof. The proof will be by induction. At $n = 1$, we have

$$\begin{aligned} T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 \\ 0 + 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \end{aligned}$$

Assuming that the statement for $n = k$, which is

$$T^k\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix},$$

is true, we will prove the statement for $n = k + 1$:

$$\begin{aligned} T^{k+1}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= T\left(T^k\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right) \\ &= T\left(\begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}\right) \\ &= \begin{bmatrix} F_{k+1} \\ F_k + F_{k+1} \end{bmatrix} \\ &= \begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix}. \end{aligned}$$

This completes our proof by induction. □

(2) Find the eigenvalues of T.

Proof. If we assume that $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of T, then there exists some eigenvalue $\lambda \in \mathbb{R}$ such that

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

So we have

$$\begin{aligned} \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} \\ &= T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= \begin{bmatrix} y \\ x+y \end{bmatrix}, \end{aligned}$$

or in other words, the system of equations

$$\begin{aligned} \lambda x &= y \\ \lambda y &= x + y. \end{aligned}$$

We can algebraically rearrange the second equation $\lambda y = x + y$ to get $(\lambda - 1)y = x$, which implies

$$\begin{aligned} (\lambda^2 - \lambda - 1)y &= \lambda(\lambda - 1)y - y \\ &= \lambda x - y \\ &= y - y \\ &= 0. \end{aligned}$$

We claim $y \neq 0$; otherwise, if $y = 0$, then the second equation $\lambda y = x + y$ will force $x = 0$, and so we would have $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, contradicting our assumption that $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector of T. Since we proved our claim $y \neq 0$, the equation $(\lambda^2 - \lambda - 1)y = 0$ implies $\lambda^2 - \lambda - 1 = 0$, from which we obtain the eigenvalues $\lambda_1 = \frac{1-\sqrt{5}}{2}$, $\lambda_2 = \frac{1+\sqrt{5}}{2}$. □

(3) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

Proof. For $\lambda_1 = \frac{1-\sqrt{5}}{2}$, we have

$$\begin{aligned} \begin{bmatrix} \frac{1-\sqrt{5}}{2}x \\ \frac{1-\sqrt{5}}{2}y \end{bmatrix} &= \frac{1-\sqrt{5}}{2} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= \begin{bmatrix} y \\ x+y \end{bmatrix}, \end{aligned}$$

from which we can equate the entries to construct the system of equations

$$\begin{aligned} \frac{1-\sqrt{5}}{2}x &= y \\ \frac{1-\sqrt{5}}{2}y &= x + y. \end{aligned}$$

So a solution can be $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{1-\sqrt{5}}{2}x \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}\right\}$, and so we can choose $v_1 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$ to be an eigenvector for $\lambda_1 = \frac{1-\sqrt{5}}{2}$. By a similar argument, we can choose $v_2 = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$ to be an eigenvector for the eigenvalue $\lambda_2 = \frac{1+\sqrt{5}}{2}$. So we can choose a basis of eigenvectors to be $\mathcal{B} = \left\{v_1 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}\right\} \subset \mathbb{R}^2$. □

(4) Use the basis from part (3) to compute $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

for each positive integer n.

Proof. Since $\mathcal{B} \subset \mathbb{R}^2$ is a basis of eigenvectors, we can write $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ as a linear combination of the eigenvectors as follows:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 1 \end{bmatrix} &= a_1 v_1 + a_2 v_2 \\
&= a_1 \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} \\
&= \begin{bmatrix} a_1 + a_2 \\ \frac{1-\sqrt{5}}{2} a_1 + \frac{1+\sqrt{5}}{2} a_2 \end{bmatrix},
\end{aligned}$$

from which we can equate the entries to construct the system of equations

$$\begin{aligned}
0 &= a_1 + a_2 \\
1 &= \frac{1-\sqrt{5}}{2} a_1 + \frac{1+\sqrt{5}}{2} a_2.
\end{aligned}$$

We can apply first equation $0 = a_1 + a_2$, or equivalently $a_2 = -a_1$, to the second equation $1 = \frac{1-\sqrt{5}}{2} a_1 + \frac{1+\sqrt{5}}{2} a_2$ to obtain

$$\begin{aligned}
1 &= \frac{1-\sqrt{5}}{2} a_1 + \frac{1+\sqrt{5}}{2} a_2 \\
&= \frac{1-\sqrt{5}}{2} a_1 - \frac{1+\sqrt{5}}{2} a_1 \\
&= \frac{(1-\sqrt{5}) - (1+\sqrt{5})}{2} a_1 \\
&= -\sqrt{5} a_1,
\end{aligned}$$

from which we get $a_1 = -\frac{1}{\sqrt{5}}$, followed immediately by $a_2 = \frac{1}{\sqrt{5}}$. Therefore, we have

$$\begin{aligned}
T^n \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) &= T^n (a_1 v_1 + a_2 v_2) \\
&= a_1 T^n v_1 + a_2 T^n v_2 \\
&= a_1 \lambda_1^n v_1 + a_2 \lambda_2^n v_2 \\
&= \left(-\frac{1}{\sqrt{5}} \right) \left(\frac{1-\sqrt{5}}{2} \right)^n \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} + \left(\frac{1}{\sqrt{5}} \right) \left(\frac{1+\sqrt{5}}{2} \right)^n \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \\ \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right) \end{bmatrix}.
\end{aligned}$$

Since $T^n \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$ at the same time, we obtain

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \\ \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right) \end{bmatrix},$$

from which we can equate the top entries to conlude

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right),$$

as desired. □