5. Solutions to Exercise 5

Exercise 5.7. Check whether the following matrices are diagonalizable. Note that you do NOT need compute the diagonalized matrix or the change-of-basis matrix.

(1) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{bmatrix}$.

Solution 5.7. A is upper-triangular matrix. Then its diagonal consists of all its eigenvalues, which are $2, 5, 8$. Since all are distinct, A is diagonalizable.

(2) $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution 5.7. The equation $\det(B - \lambda I) = 0$ is $(1-\lambda)(4-\lambda) - 6 = 0$. The equation is $\lambda^2 - 5\lambda - 2 = 0$, which has two distinct solutions since $(-5)^2 - 4 \times 1 \times (-2) \neq 0$. Then B has two distinct eigenvalues. Then it is diagonalizable.

(3) $C = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution 5.7. The equation $\det(C - \lambda I) = 0$ is $(2-\lambda)^2(3-\lambda) = 0$. It has two solutions 2 and 3 where 2 is a double solution.

- For eigenvalue $\lambda = 2$, the eigenspace is the solution to the equation $(C - 2I)X = 0$ for $X \in \mathbb{C}^3$:

$$\begin{bmatrix} 2-2 & -1 & 0 \\ 0 & 3-2 & 0 \\ 1 & 1 & 2-2 \end{bmatrix} X = 0.$$

  The solution is $X \in \texttt{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

- For eigenvalue $\lambda = 3$, the eigenspace is the solution to the equation $(C - 3I)X = 0$ for $X \in \mathbb{C}^3$:
$$\begin{bmatrix} 2-3 & -1 & 0 \\ 0 & 3-3 & 0 \\ 1 & 1 & 2-3 \end{bmatrix} X = 0.$$
The solution is $X \in \texttt{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$.

Then we can only find two linearly independent eigenvectors. Therefore C is not diagonalizable.

Exercise 5.8.

(1) Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \texttt{Nul}(T) \oplus \texttt{im}(T)$.

Solution 5.8. Since T is diagonalizable, then there exists a basis of V $\{v_1, \ldots, v_n\}$ consisting of eigenvectors. Then there exists $\lambda \in \mathbb{F}$ such that $T(v_i) = \lambda_i v_i$ for $i = 1, \ldots, n$. Without loss of generality, we may assume that $\lambda_1 = \ldots = \lambda_k = 0$ and $\lambda_{k+1}, \ldots, \lambda_n \neq 0$. Then $v_1, \ldots, v_k \in \texttt{Nul}(T)$, and $v_{k+1}, \ldots, v_n \in \texttt{im}(T)$. So since $V = \texttt{Span}(v_1, \ldots, v_k) \oplus \texttt{Span}(v_{k+1}, \ldots, v_n) = \texttt{Nul}(T) \oplus \texttt{im}(T)$.

(2) State the converse of the statement above. Prove it or give a counterexample.

Solution 5.8. The converse is "If $V = \texttt{Nul}(T) \oplus \texttt{im}(T)$, then T is diagonalizable." $V = \mathbb{C}^3$ and T being defined by $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is a counter example. Here $\texttt{Nul}(T) = \texttt{Span}(e_1)$, and $\texttt{im}(T) = \texttt{Span}(e_2, e_3)$ but A is not diagonalizable.

Exercise 5.9. Give an example that $R, T \in \mathcal{L}(\mathbb{C}^4)$ such that R and T each have $2, 6, 7$ as eigenvalues and no other eigenvalues, and there does not exist an invertible operator $S \in \mathcal{L}(\mathbb{C}^4)$ such that $R = S^{-1}TS$.

Solution 5.9. Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ and $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$ be two bases. Let R be defined by $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ under the basis E and T be defined by $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ under the basis $\mathcal{F}$. Assume such an invertible operator S exists. Then let $v_1 = S^{-1}(f_1)$ and $v_2 = S^{-1}(f_2)$. Since $\{f_1, f_2\}$ is linearly independent, $\{v_1, v_2\}$ should be linearly independent. Since

$$R(v_1) = S^{-1}TS(v_1) = S^{-1}T(f_1) = S^{-1}(2f_1) = 2S^{-1}(f_1) = 2v_1,$$

$$R(v_2) = S^{-1}TS(v_2) = S^{-1}T(f_2) = S^{-1}(2f_2) = 2S^{-1}(f_2) = 2v_2,$$

v_1 and v_2 are linearly independent eigenvectors of R corresponding to the eigenvalue 2, which is impossible. Therefore the assumption is wrong. Then such an invertible operator doesn't exist.

Exercise 5.10. Let V be finite-dimensional, and $T, S \in \mathcal{L}(V)$. Suppose T has $\dim V$ distinct eigenvalues, and S has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

Solution 5.10. Since T have $\dim V$ distinct eigenvalues, the eigenvectors of T corresponding to distinct eigenvalues form a basis. Let $v_1, \ldots, v_n$ be these eigenvectors and $\lambda_1, \ldots, \lambda_n$ be corresponding eigenvalues. Then we have $T(v_1) = \lambda_1 v_1, \ldots, T(v_n) = \lambda_n v_n$.

S has the same eigenvectors as T, and we denote the corresponding eigenvalues by $\mu_1, \ldots, \mu_n$. Then we have $S(v_1) = \mu_1 v_1, \ldots, S(v_n) = \mu_n v_n$.

For any $v \in V$, since $\{v_1, \ldots, v_n\}$ is a basis of V, $v = c_1 v_1 + \ldots + c_n v_n$. Then

$$TS(v) = TS(c_1v_1 + \ldots + c_nv_n) = T(c_1S(v_1) + \ldots + c_nS(v_n)) = T(c_1\mu_1v_1 + \ldots + c_n\mu_nv_n)$$

$$= c_1\mu_1T(v_1) + \ldots + c_n\mu_nT(v_n) = c_1\mu_1\lambda_1v_1 + \ldots + c_n\mu_n\lambda_nv_n,$$

and

$$ST(v) = ST(c_1v_1 + \ldots + c_nv_n) = S(c_1T(v_1) + \ldots + c_nT(v_n)) = S(c_1\lambda_1v_1 + \ldots + c_n\lambda_nv_n)$$

$$= c_1\lambda_1S(v_1) + \ldots + c_n\lambda_nS(v_n) = c_1\lambda_1\mu_1v_1 + \ldots + c_n\lambda_n\mu_nv_n.$$

So $TS(v) = ST(v)$. Since this holds for any $v \in V$, $ST = TS$.

Exercise 5.11. The **Fibonacci sequence** $F_1, F_2, \ldots$ is defined by

$$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 3.$$

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x+y \end{bmatrix}.$$

(1) Show that $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$ for each positive integer n.

Solution 5.11. Use induction. Let $P(n) =$ "$T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$" for each positive integer n.

Base case $n = 1$: $T^1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$.

Induction Step: Assume that $P(k)$ is true for some $k \geq 1$. Then $T^k\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}$ for some $k \geq 1$. Then

$$T^{k+1}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(T^k\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right) = T\left(\begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}\right) = \begin{bmatrix} F_{n+1} \\ F_n + F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix}.$$

Then $P(k+1)$ is true.

By induction, $P(n)$ is true for any $n \geq 1$. Then $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$ for each positive integer n.

(2) Find the eigenvalues of T.

Solution 5.11. Let $\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$. Then since

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2,$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e_1 + e_2,$$

The matrix of T is

$$A = \begin{bmatrix} T \end{bmatrix}_{\mathcal{S}\leftarrow\mathcal{S}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

To find the eigenvalues, we need to solve the equation $\det(A - \lambda I) = 0$. That is $\lambda^2 - \lambda - 1 = 0$. So eigenvalues are $\dfrac{1 \pm \sqrt{5}}{2}$.

(3) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

Solution 5.11. Consider the two eigenvalues λ one by one.

$\lambda = \dfrac{1+\sqrt{5}}{2}$**:** Solve the equation $(A - \dfrac{1+\sqrt{5}}{2} I)X = 0$ for $X \in \mathbb{R}^2$. Solutions are

$$X \in \texttt{Span}\left(\begin{bmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{bmatrix}\right).$$

$\lambda = \dfrac{1-\sqrt{5}}{2}$**:** Solve the equation $(A - \dfrac{1-\sqrt{5}}{2} I)X = 0$ for $X \in \mathbb{R}^2$. Solutions are

$$X \in \texttt{Span}\left(\begin{bmatrix} \frac{-\sqrt{5}-1}{2} \\ 1 \end{bmatrix}\right).$$

Then $\mathcal{B} = \left\{ v_1 = \begin{bmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} \frac{-\sqrt{5}-1}{2} \\ 1 \end{bmatrix} \right\}$ is a basis consisting of eigenvectors of T. Then the matrix under this basis is

$$\begin{bmatrix} T \end{bmatrix}_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

(4) Use the basis from part (c) to compute $T^n\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each positive integer n.

Solution 5.11. Since $\mathcal{B} = \left\{ v_1 = \begin{bmatrix}\frac{\sqrt{5}-1}{2}\\1\end{bmatrix}, v_2 = \begin{bmatrix}\frac{-\sqrt{5}-1}{2}\\1\end{bmatrix}\right\}$ is a basis, the change-of-basis matrix

$$P_{\mathcal{S}\leftarrow\mathcal{B}} = \begin{bmatrix}\frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2}\\1 & 1\end{bmatrix}.$$

Then

$$\left[e_2\right]_{\mathcal{B}} = P_{\mathcal{S}\leftarrow\mathcal{B}}^{-1}\left[e_2\right]_S = \begin{bmatrix}\frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2}\\1 & 1\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}\frac{\sqrt{5}+1}{2}\\\frac{\sqrt{5}-1}{2}\end{bmatrix}.$$

Then

$$\left[T^n\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right]_{\mathcal{B}} = \left[T^n(e_2)\right]_{\mathcal{B}} = \left[T^n\right]_{\mathcal{B}\leftarrow\mathcal{B}}\left[e_2\right]_{\mathcal{B}} = \left[T\right]^n_{\mathcal{B}\leftarrow\mathcal{B}}\left[e_2\right]_{\mathcal{B}} = \begin{bmatrix}\frac{1+\sqrt{5}}{2} & 0\\0 & \frac{1-\sqrt{5}}{2}\end{bmatrix}^n \frac{1}{\sqrt{5}}\begin{bmatrix}\frac{\sqrt{5}+1}{2}\\\frac{\sqrt{5}-1}{2}\end{bmatrix}$$

$$= \begin{bmatrix}\left(\frac{1+\sqrt{5}}{2}\right)^n & 0\\0 & \left(\frac{1-\sqrt{5}}{2}\right)^n\end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix}\frac{\sqrt{5}+1}{2}\\\frac{\sqrt{5}-1}{2}\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\\-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\end{bmatrix}.$$

Then

$$T^n\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} v_1 - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} v_2\right)$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\begin{bmatrix}\frac{\sqrt{5}-1}{2}\\1\end{bmatrix} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\begin{bmatrix}\frac{-\sqrt{5}-1}{2}\\1\end{bmatrix}\right)$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix}\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\\\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\end{bmatrix}.$$

Since $T^n\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}F_n\\F_{n+1}\end{bmatrix}$, then

$$F_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$