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MATH 132 Homework 5

8.1. Let V be a finite-dimensional complex vector space of dimension n, and $T \in \mathcal{L}(V)$. Please show that $\operatorname{nul} T^n$ and $\operatorname{im} T^n$ are invariant under T.

Proof. If $v \in \operatorname{nul} T^n$ be arbitrary, then $T^n v = 0$, which implies $T^n(Tv) = T(T^n v) = T(0) = 0$, and so $Tv \in \operatorname{nul} T^n$, which means $\operatorname{nul} T^n$ is invariant under T. If $w \in \operatorname{im} T^n$ is arbitrary, then there exists $u \in V$ such that $T^n u = w$, which implies $T^n(Tu) = T(T^n u) = Tw$, and so $Tw \in \operatorname{im} T^n$ since $T \in \mathcal{L}(V)$ implies $Tu \in V$, which means $\operatorname{im} T^n$ is invariant under T. □

8.2. Suppose $T \in \mathcal{L}(V)$ and m is a non-negative integer. Show that $\operatorname{nul} T^m = \operatorname{nul} T^{m+1}$ if and only if $\operatorname{im} T^m = \operatorname{im} T^{m+1}$.

Proof. We will first prove the forward direction: If $\operatorname{nul} T^m = \operatorname{nul} T^{m+1}$, then $\operatorname{im} T^m = \operatorname{im} T^{m+1}$. Since we assumed $\operatorname{nul} T^m = \operatorname{nul} T^{m+1}$, we can apply the Rank-Nullity Theorem twice to obtain

$$\begin{aligned} \dim \operatorname{im} T^m &= \dim V - \dim \operatorname{nul} T^m \\ &= \dim V - \dim \operatorname{nul} T^{m+1} \\ &= \dim \operatorname{im} T^{m+1}. \end{aligned}$$

We recall part (1) of Exercise 8.14 (instructor's additional exercise) which asserts $\operatorname{im} T^m \supseteq \operatorname{im} T^{m+1}$ for all integers $m \geq 0$. But, in order to satisfy $\dim \operatorname{im} T^m = \dim \operatorname{im} T^{m+1}$, we must have $\operatorname{im} T^m = \operatorname{im} T^{m+1}$. Now we will prove the backward direction: If $\operatorname{im} T^m = \operatorname{im} T^{m+1}$, then $\operatorname{nul} T^m = \operatorname{nul} T^{m+1}$. Since we assumed $\operatorname{im} T^m = \operatorname{im} T^{m+1}$, we can apply the Rank-Nullity Theorem twice to obtain

$$\begin{aligned} \dim \operatorname{nul} T^m &= \dim V - \dim \operatorname{im} T^m \\ &= \dim V - \dim \operatorname{im} T^{m+1} \\ &= \dim \operatorname{nul} T^{m+1}. \end{aligned}$$

We recall Lemma 8.1.1 of the Notes which asserts $\operatorname{nul} T^m \subseteq \operatorname{nul} T^{m+1}$ for all integers $m \geq 0$. But, in order to satisfy $\dim \operatorname{nul} T^m = \dim \operatorname{nul} T^{m+1}$, we must have $\operatorname{nul} T^m = \operatorname{nul} T^{m+1}$. □

8.3. Let $T \in \mathcal{L}(\mathbb{C}^2)$ be defined in the following ways. Find all the generalized eigenspaces.

(1) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} -b \\ a \end{bmatrix}$

Proof. We consider the standard basis $\mathcal{S} := \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. To find the eigenvalues of T, we can let $A = [T]_{\mathcal{S} \leftarrow \mathcal{S}}$ be the matrix representation of T with respect to $\mathcal{S}$ and solve the equation $\det(A - \lambda I) = 0$; that is, we have

$$\begin{aligned} A &= \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} \\ &= \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \end{aligned}$$

and so

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) \\ &= \lambda^2 + 1 \\ &= (\lambda + i)(\lambda - i), \end{aligned}$$

from which we obtain the eigenvalues $\lambda = i, -i$. As we have two distinct eigenvalues, there exist at least two linearly independent eigenvectors corresponding to their respective eigenvalues, and so there must be at least two eigenspaces. Remark 8.2.4 of the Notes implies in particular that the eigenvectors for $\lambda = i, -i$ are examples of generalized eigenvectors for $\lambda = i, -i$. Note that we can have no more than two eigenspaces in $\mathbb{C}^2$; otherwise, if we had more than two eigenspaces in $\mathbb{C}^2$, then the basis of $\mathbb{C}^2$ would contain at least three linearly independent vectors, implying that $\dim \mathbb{C}^2 \geq 3$, which is a contradiction. So we have exactly two eigenspaces, and furthermore all generalized eigenspaces V_i^G, V_{-i}^G are eigenspaces V_i, V_{-i}. To say this more explicitly: for $\lambda = i$ we can solve

$$\begin{aligned} 0 &= (A - iI)x \\ &= \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

to obtain $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ -i \end{bmatrix}\right\} = V_i = V_i^G$, and for $\lambda = -i$ we can solve

$$\begin{aligned} 0 &= (A - (-i)I)x \\ &= \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

to obtain $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ ix_1 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ i \end{bmatrix}\right\} = V_{-i} = V_{-i}^G$. □

(2) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} b \\ 0 \end{bmatrix}$

Proof. We consider the standard basis $\mathcal{S} := \left\{e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$. To find the eigenvalues of T, we can let $A = [T]_{\mathcal{S}\leftarrow\mathcal{S}}$ be the matrix representation of T with respect to $\mathcal{S}$ and solve the equation $\det(A - \lambda I) = 0$; that is, we have

$$\begin{aligned} A &= \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} \\ &= \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

and so

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) \\ &= \lambda^2, \end{aligned}$$

from which we obtain the eigenvalue $\lambda = 0$ of multiplicity 2, which implies that there exists a generalized eigenspace V_0^G of dimension 2. By Theorem 8.2.5 of the Notes, we have $V_0^G = \operatorname{nul}(A - 0I)^2$. To find elements of $\operatorname{nul}(A - 0I)^2$ is equivalent to solving $(A - 0I)^2 = 0$. So we have

$$\begin{aligned} 0 &= (A - 0I)^2 x \\ &= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)^2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

So we have $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \operatorname{span}\{e_1, e_2\} = V$. Altogether, we have

$$\begin{aligned} V_0^G &= \operatorname{nul}(A - 0I)^2 \\ &= \operatorname{span}\{e_1, e_2\} \\ &= V; \end{aligned}$$

in other words, $V_0^G = V$ is the generalized eigenspace for $\lambda = 0$. □

8.4. Prove or give a counterexample: If V is a complex vector space and $\dim V = n$ and $T \in \mathcal{L}(V)$, then T^n is diagonalizable.

Proof. Counterexample: Let $V = \mathbb{C}^2$ so that V is a complex vector space with $\dim V = 2$, and let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

be the matrix representation of T so that $T \in \mathcal{L}(V)$. Then

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

is a matrix representation of T^2. But the only eigenvalue of A^2 is $\lambda = 1$, which produces only one linearly independent eigenvector such as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So there does not exist a basis of eigenvectors of T^2 in $\mathbb{C}^2$, and so Theorem 5.5.2 of the Notes implies that T^2 is not diagonalizable. □

8.5. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T.

Proof. We will first prove the forward direction: If V has a basis consisting of eigenvectors of T, then every generalized eigenvector of T is an eigenvector of T. Let $v \in V_\lambda^G$, which means v is a generalized eigenvector for the eigenvalue λ. We note that generalized eigenspaces such as V_λ^G intersect other eigenspaces for different eigenvalues not equal to λ at $\{0\}$, which implies that v will not be a scalar multiple of the eigenvectors of the eigenspaces that intersect V_λ^G trivially. So v is only a linear combination of eigenvectors for our eigenvalue λ, which means $v \in V_\lambda$. Therefore, $V_\lambda^G \subset V_\lambda$, meaning every generalized eigenvector of T is an eigenvector of T. (Of course, we also have $V_\lambda \subset V_\lambda^G$ because clearly every eigenvector of T is also a generalized eigenvector of T, so we really have $V_\lambda = V_\lambda^G$, but that is technically not too relevant here.)

Now we will prove the backward direction: If every generalized eigenvector of T is an eigenvector of T, then V has a basis consisting of eigenvectors of T. Since we assumed that every generalized eigenvector of T is an eigenvector of T, it follows that every generalized eigenspace of T is an eigenspace of T. We also know by Theorem 8.3.2 of the Notes that V is a direct sum of generalized eigenspaces, which means V is also a direct sum of eigenspaces. By Theorem 5.5.2 of the Notes, T is diaongalizable; equivalently, V has a basis consisting of eigenvectors of T. □