Solutions to Exercise 8

Exercise 8.1. Let V be a finite-dimensional complex vector space of dimension n, and $T \in \mathcal{L}(V)$. Please show that $\texttt{Nul}(T^n)$ and $\texttt{im}(T^n)$ are all invariant under T.

Solution 8.1. For any $v \in \texttt{Nul}(T^n)$, $T^n(v) = 0$. Then $T^n(T(v)) = T(T^n(v)) = T(0) = 0 \in \texttt{Nul}(T^n)$. So $\texttt{Nul}(T^n)$ is invariant under T.

For any $v \in \texttt{im}(T^n)$, $\exists w \in V$ such that $v = T^n(w)$. Then $T(v) = T(T^n(w)) = T^n(T(w)) \in \texttt{im}(T^n)$. So $\texttt{im}(T^n)$ is invariant under T.

Exercise 8.2. Suppose $T \in \mathcal{L}(V)$ and m is a non-negative integer. Show that $\texttt{Nul}(T^m) = \texttt{Nul}(T^{m+1})$ if and only if $\texttt{im}(T^m) = \texttt{im}(T^{m+1})$.

Solution 8.2. Let $\dim V = n$. For any $v \in \texttt{im}(T^{k+1})$, there exists $w \in V$ such that $v = T^{k+1}(w)$. Then $v = T^k(T(w))$. So $v \in \texttt{im}(T^k)$. So $\texttt{im}(T^k) \supset \texttt{im}(T^{k+1})$ for any $k \geq 0$.

(⇒): Assume that $\texttt{Nul}(T^m) = \texttt{Nul}(T^{m+1})$. Then

$$\dim \texttt{im}(T^m) = n - \dim \texttt{Nul}(T^m) = n - \dim \texttt{Nul}(T^{m+1}) = \dim \texttt{im}(T^{m+1}).$$

Since $\texttt{im}(T^m) \supset \texttt{im}(T^{m+1})$, we have that $\texttt{im}(T^m) = \texttt{im}(T^{m+1})$.

(⇐): Assume that $\texttt{im}(T^m) = \texttt{im}(T^{m+1})$. Then

$$\dim \texttt{Nul}(T^m) = n - \dim \texttt{im}(T^m) = n - \dim \texttt{im}(T^{m+1}) = \dim \texttt{Nul}(T^{m+1}).$$

Since $\texttt{Nul}(T^m) \subset \texttt{Nul}(T^{m+1})$, we have that $\texttt{Nul}(T^m) = \texttt{Nul}(T^{m+1})$.

Exercise 8.3. Let $T \in \mathcal{L}(\mathbb{C}^2)$ be defined in the following ways. Find all the generalized eigenspaces.

(1) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} -b \\ a \end{bmatrix}$.

Solution 8.3. Let $V = \mathbb{C}^2$. Consider the basis $\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then let $A = \begin{bmatrix} T \end{bmatrix}_{\mathcal{S} \leftarrow \mathcal{S}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. To find eigenvalues, we need to solve the equation $\det(A - \lambda I) = 0$. The equation is $\lambda^2 + 1 = 0$. The solutions are $\pm i$. Since there are two eigenvalues, there should be two eigenspaces and then all generalized eigenspaces are eigenspaces.

$\lambda = i$**:** Solve the equation $(A - iI)X = 0$:

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} X = 0.$$

The solutions are $X \in \texttt{Span}\left(\begin{bmatrix} i \\ 1 \end{bmatrix} \right)$. This is $V_i^G = V_i = \texttt{Span}\left(\begin{bmatrix} i \\ 1 \end{bmatrix} \right)$.

$\lambda = -i$**:** Solve the equation $(A - (-i)I)X = 0$:

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} X = 0.$$

The solutions are $X \in \texttt{Span}\left(\begin{bmatrix} -i \\ 1 \end{bmatrix} \right)$. This is $V_{-i}^G = V_{-i} = \texttt{Span}\left(\begin{bmatrix} -i \\ 1 \end{bmatrix} \right)$.

Then there are two generalized eigenspaces, which are all eigenspaces. They are listed above.

(2) $T\left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} b \\ 0 \end{bmatrix}$.

Solution 8.3. Let $V = \mathbb{C}^2$. Consider the basis $\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then let $A = \begin{bmatrix} T \end{bmatrix}_{\mathcal{S} \leftarrow \mathcal{S}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To find eigenvalues, we need to solve the equation $\det(A - \lambda I) = 0$. The equation is $\lambda^2 = 0$. The only solution is 0. Therefore there is a generalized eigenspace of dimension 2 corresponding to the eigenvalue 0. Then since $V_0^G = \texttt{Nul}\left((A - 0I)^2\right)$, we need to solve the equation $(A - 0I)^2 X = 0$. That is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} X = 0.$$

The solutions are $X \in \texttt{Span}\,(e_1, e_2) = V$. Then the generalized eigenspace is $V_0^G = V$.

Exercise 8.4. Prove or give a counterexample: If V is a complex vector space and $\dim V = n$ and $T \in \mathcal{L}(V)$, then T^n is diagonalizable.

Solution 8.4. Counterexample: $V = \mathbb{C}^2$ and T is defined by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $T^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ which is not diagonalizable.

Exercise 8.5. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T.

Solution 8.5.

(⇒): Suppose that V has a basis consisting of eigenvectors of T. Let $v \in V_\lambda^G$ be a generalized eigenvector corresponding to the eigenvalue λ. Since generalized eigenspaces with respect to different eigenvalues intersect at $\{0\}$, then V_λ^G intersect with other eigenspaces whose eigenvalues $\neq \lambda$ only at $\{0\}$. Then when writing the coordinate of v under the basis consisting of eigenvectors, the eigenvectors of eigenvalues $\neq \lambda$ contribute nothing. Therefore v is a linear combination of eigenvectors corresponding to the eigenvalue λ. Then $v \in V_\lambda$. Then $V_\lambda^G = V_\lambda$. Therefore any generalized eigenvectors are also eigenvectors.

(⇐): Since every generalized eigenvector of T is an eigenvector, then every generalized eigenspace is an eigenspace. Since V is a direct sum of generlized eigenspaces, it is a direct sum of eigenspaces. Then V has a basis consisting of eigenvectors of T by Conditions equivalent to diagonalizability.