MATH 132 discussion
University of California, Riverside

Ryan Ta
Spring 2019

MATH 132 Homework 6

8.6. Suppose $S, T \in \mathcal{L}(V)$ and ST is nilpotent. Prove that TS is nilpotent.

Proof. Since ST is nilpotent, there exists some positive integer i such that $(ST)^i = 0$. So, using associativity of linear maps, we have

$$\begin{aligned}(TS)^{i+1} &= \underbrace{(TS)(TS)\cdots(TS)(TS)}_{i+1 \text{ terms}} \\ &= T\underbrace{(ST)\cdots(ST)}_{i \text{ terms}}S \\ &= T(ST)^i S \\ &= T0S \\ &= 0,\end{aligned}$$

and so TS is nilpotent. □

8.7. Prove or give a counterexample: The set of nilpotent operators on V is a subspace of $\mathcal{L}(V)$.

Proof. Disprove: Define S, T by their matrix representations

$$\begin{aligned}A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\end{aligned}$$

respectively. Then $A^2 = B^2 = 0$, which means S, T are nilpotent. But at the same time we have

$$A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, for all positive integers i, we have $(A+B)^i = I$ if i is even and $(A+B)^i = A + B$ if i is odd. In other words, $(A+B)^i \neq 0$ for all positive integers i, which means $S + T$ is not nilpotent. This violates the closure property of a subspace, and so the set of nilpotent operators on V is not a subspace of $\mathcal{L}(V)$. □

8.8. Give an example of an operator T on a finite-dimensional real vector space such that 0 is the only eigenvalue of T but T is not nilpotent.

Proof. Let $V = \mathbb{R}^3$, let $\mathcal{B} \subset \mathbb{R}^3$ be a basis and define $A = [T]_{\mathcal{B} \leftarrow \mathcal{B}}$ by

$$A := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then solving $\det(A - \lambda I) = 0$ shows that the only real (that is, non-imaginary) eigenvalue of A is 0. However, we have

$$\begin{aligned}A^2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ A^3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \\ A^4 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ A^5 &= A,\end{aligned}$$

all of which are clearly nonzero matrices. This means for all integers $i \geq 5$ that A^i is a product of A, A^2, A^3, A^4 and hence nonzero. This is enough to demonstrate that we have $A^i \neq 0$, and therefore $T^i \neq 0$, for all positive integers i, and so T is not nilpotent. □

8.9. Suppose $T \in \mathcal{L}(\mathbb{C}^4)$ is such that the eigenvalues of T are $3, 5, 8$. Prove that $(T-3I)^2(T-5I)^2(T-8I)^2 = 0$.

Proof. We have that $3, 5, 8$ are the only three distinct eigenvalues of T and the size of the matrix representation of T is 4. Since this matrix representation of T is similar to the corresponding Jordan canonical form of T, we can determine that the largest Jordan block is 2×2. This implies the highest degree of a factor of a minimal polynomial is 2. So the minimal polynomial of T is exactly one of $(x-3)^2(x-5)(x-8), (x-3)(x-5)^2(x-8), (x-3)(x-5)(x-8)^2$; in other words, we have exactly one of $(T-3I)^2(T-5I)(T-8I) = 0, (T-3I)(T-5I)^2(T-8I) = 0, (T-3I)(T-5I)(T-8I)^2 = 0$, which means in all three cases we get $(T-3I)^2(T-5I)^2(T-8I)^2 = 0$.

Alternate proof: Theorem 8.3.2 of the Notes asserts that V is a direct sum of generalized eigenspaces; that is, $\mathbb{C}^4 = V_3^G \oplus V_5^G \oplus V_8^G$. Taking dimensions, we get $4 = \dim C^4 = \dim V_3^G + \dim V_5^G + \dim V_8^G$. This implies that one of V_3^G, V_5^G, V_8^G must have dimension 2 and the other two have dimension 1; in other words one of the eigenvalues $3, 5, 8$ have multiplicity 2 and the other two multiplicity 1. For instance, if 3 has multiplicity 2, then there exists a generalized eigenvector $x \in \mathbb{C}^4$ satisfying $(T-3I)^2 x = 0$, which implies $(T-3I)^2 = 0$. In any case, we have exactly one of $(T-3I)^2 = 0, (T-5I)^2 = 0, (T-8I)^2 = 0$, which means in all three cases we get $(T-3I)^2(T-5I)^2(T-8I)^2 = 0$. □

8.10. (1) Give an example of an operator on $\mathbb{C}^4$ whose characteristic polynomial equals $(x-1)(x-3)^3$ and whose minimal polynomial equals $(x-1)(x-3)^2$.

Proof. Let $\mathcal{B} \subset \mathbb{C}^4$ be a basis and define $A = [T]_{\mathcal{B}\leftarrow\mathcal{B}}$ by

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Then, by Definition 8.4.2 of the Notes, the characteristic polynomial of A is

$$\begin{aligned} \det(A - xI) &= \det\left(\begin{bmatrix} 1-x & 0 & 0 & 0 \\ 0 & 3-x & 1 & 0 \\ 0 & 0 & 3-x & 0 \\ 0 & 0 & 0 & 3-x \end{bmatrix}\right) \\ &= (1-x)(3-x)^3 \\ &= (x-1)(x-3)^3. \end{aligned}$$

To find the minimal polynomial of A, we evaluate all the factors of our characteristic polynomial of A. We have

$$\begin{aligned} (A-I)(A-3I) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\neq 0, \end{aligned}$$

which means $(x-1)(x-3)$ is not the minimal polynomial of A. This implies that all factors of $(x-1)(x-3)$—that is, $x-1, x-3$—are also not minimal polynomials of A; otherwise, if either $x-1$ or $x-3$ were, then we would have $A - I = 0$ or $A - 3I = 0$, either of which would imply $(A-I)(A-3I) = 0$ which contradicts $(A-I)(A-3I) \neq 0$. However, we have

$$\begin{aligned} (A-I)(A-3I)^2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= 0. \end{aligned}$$

So the minimal polynomial of A is $(x-1)(x-2)^2$. □

(2) Give an example of an operator on $\mathbb{C}^4$ whose characteristic and minimal polynomials both equal $x(x-1)^2(x-3)$.

Proof. Let $C \subset \mathbb{C}^4$ be a basis and define $B = [T]_{C \leftarrow C}$ by

$$B := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Then, by Definition 8.4.2 of the Notes, the characteristic polynomial of B is

$$\begin{aligned} \det(B - xI) &= \det\left(\begin{bmatrix} -x & 0 & 0 & 0 \\ 0 & 1-x & 1 & 0 \\ 0 & 0 & 1-x & 0 \\ 0 & 0 & 0 & 3-x \end{bmatrix}\right) \\ &= -x(1-x)^2(3-x) \\ &= x(x-1)^2(x-3). \end{aligned}$$

To show that the minimal polynomial of B is our characteristic polynomial of B, we need to show that all the factors of our characteristic polynomial of B of lower degree is not the minimal polynomial of B. We have

$$\begin{aligned} B(B-I)(B-3I) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\neq 0, \end{aligned}$$

which means $x(x-1)(x-3)$ is not the minimal polynomial of A. This implies that all the factors of lower degree—that is, $x, x-1, x-3, x(x-1), x(x-3), (x-1)(x-3)$—are also not minimal polynomials of B; otherwise, if any one of them were, then we would have one of $B = 0, B - I = 0, B - 3I = 0, B(B-I) = 0, B(B-3I) = 0, (B-I)(B-3I) = 0$, any one of which would imply $B(B-I)(B-3I) = 0$ which contradicts $B(B-I)(B-3I) \neq 0$. Hence, none of $x, x-1, x-3, x(x-1), x(x-3), (x-1)(x-3), x(x-1)(x-3)$ are minimal polynomials of B, which means $x(x-1)^2(x-3)$ must be the minimal polynomial of B and therefore coincide with the characteristic polynomial of B. □

8.11. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if the minimal polynomial of T has no repeated zeros.

Proof. (I am stacking "if and only if" statements instead of proving the forward and backward directions separately.) Theorem 5.5.2 of the Notes asserts that V has a basis consisting of eigenvectors of T if and only if T is diagonalizable. And T is diagonalizable if and only if the largest Jordan block corresponding to any eigenvalue of T is size 1×1. Finally, Remark 8.4.6 of the Notes asserts that each factor $(x - \lambda_i)^{r_i}$ of a minimal polynomial $(x - \lambda_1)^{r_1} \cdots (x - \lambda_m)^{r_m}$ is related to a Jordan block J_{λ_i} of the size r_i. So each Jordan block is size 1×1 if and only if the minimal polynomial of T has no repeated zeros. □

8.12. Let

$$N := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Please find the characteristic polynomial and minimal polynomial of N.

Proof. By Definition 8.4.2 of the Notes, the characteristic polynomial of N is

$$\begin{aligned} \det(N - xI) &= \det\left(\begin{bmatrix} -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & 1 \\ 0 & 0 & 0 & -x \end{bmatrix}\right) \\ &= x^4. \end{aligned}$$

Also, we have

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and so the minimal polynomial of N is x^4, which coincides with the characteristic polynomial of N. (Another way of determining that the minimal polynomial of N is x^4 is realizing that $N - xI$ is already a Jordan block of size 4×4.) □

8.13. Let

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Please find its Jordan canonical form C and find the transformation matrix P such that $C = P^{-1}AP$.

Proof. Our characteristic polynomial of A is

$$\begin{aligned} \det(A - xI) &= \det\left(\begin{bmatrix} 1-x & 2 & 3 \\ 0 & 1-x & 2 \\ 0 & 0 & 2-x \end{bmatrix}\right) \\ &= (1-x)^2(2-x). \end{aligned}$$

Next, we will find the minimal polynomial of A. Since

$$\begin{aligned} (I - A)(2I - A) &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\right)\left(2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & -2 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\neq 0, \end{aligned}$$

it follows that $(1-x)(2-x)$ is not a minimal polynomial of A. This also implies that $1-x, 2-x$ are not minimal polynomials of A. Therefore, the minimal polynomial of A is $(1-x)^2(2-x)$, which coincides with the characteristic polynomial of A. Following Remark 8.4.6 of the Notes, which asserts that each factor $(x-\lambda_i)^{r_i}$ of a minimal polynomial $(x-\lambda_1)^{r_1}\cdots(x-\lambda_m)^{r_m}$ is related to a Jordan block J_{λ_i} of the size r_i, our Jordan blocks corresponding to our eigenvalues $1, 2$ are

$$\begin{aligned} J_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \\ J_2 &= \begin{bmatrix} 2 \end{bmatrix}, \end{aligned}$$

and so the Jordan canonical form of A is

$$\begin{aligned} C &= \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \end{aligned}$$

Next, we will find a transformation matrix P satisfying $C = P^{-1}AP$. For the eigenvalue 1, we solve $(A - I)x = 0$ to get

$$\begin{aligned} 0 &= (A - I)x \\ &= \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \end{aligned}$$

from which we obtain the eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$, and so we can choose our first basis vector to be $v_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Next, we solve $(A - I)x = v_1$ to get

$$\begin{aligned} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= v_1 \\ &= (A - I)x \\ &= \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \end{aligned}$$

from which we obtain the eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \operatorname{span}\left\{\begin{bmatrix} 2x_1 \\ 1 \\ 0 \end{bmatrix}\right\}$, and so we can choose the representative vector to be our second basis vector $v_2 := \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$. Finally, for the eigenvalue 2, we solve $(A - 2I)x = 0$ to get

$$\begin{aligned} 0 &= (A - 2I)x \\ &= \begin{bmatrix} -1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \end{aligned}$$

from which we obtain the eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7x_3 \\ 2x_3 \\ x_3 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}\right\}$, and so we can choose our third basis vector to be $v_3 := \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$. Since the matrix representation C of T under our basis $\mathcal{B} := \{v_1, v_2, v_3\} \subset \mathbb{C}^3$ is in Jordan canonical form, it follows that $\mathcal{B}$ is a Jordan basis. This means that our transformation matrix is

$$\begin{aligned} P &= \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 7 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Furthermore,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & 0 & 7 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 7 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -7 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 7 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= C, \end{aligned}$$

as desired. □