Exercise 8.6. Suppose $S, T \in \mathcal{L}(V)$ and ST is nilpotent. Prove that TS is nilpotent.

Solution 8.6. Since ST is nilpotent, then there exists m such that $(ST)^m = 0$. Then $(TS)^{m+1} = T(ST)^m S = 0$. So TS is also nilpotent.

Exercise 8.7. Prove or give a counterexample: The set of nilpotent operators on V is a subspace of $\mathcal{L}(V)$.

Solution 8.7. Let $T, S \in \mathcal{L}(\mathbb{C}^2)$ be two operators defined by $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then since $T^2 = S^2 = 0$, both of them are nilpotent. However $T + S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which is not nilpotent. So this statement is wrong.

Exercise 8.8. Give an example of an operator T on a finite-dimensional real vector space such that 0 is the only eigenvalue of T but T is not nilpotent.

Solution 8.8. $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is one example.

Remark 8.0.1. The key point of the problem is to really understand the relation between the characteristic polynomials, eigenvalues, and nilpotency. Eigenvalues are solutions to the characteristic polynomials in the **working** field, and nilpotency requires that the solutions to the characteristic polynomials are all zeroes in the **complex** field.

Exercise 8.9. Suppose $T \in \mathcal{L}(\mathbb{C}^4)$ is such that the eigenvalues of T are 3, 5, 8. Prove that $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$.

Solution 8.9. Since there are three eigenvalues of T, and the size of T is 4, then the biggest size of Jordan block of T is 2. Then the highest degree of factors of the minimal polynomial of T is 2. Then there are only four possible minimal polynomials:

- $(x - 3)^2(x - 5)(x - 8)$,
- $(x - 3)(x - 5)^2(x - 8)$,
- $(x - 3)(x - 5)(x - 8)^2$,

- $(x-3)(x-5)(x-8)$.

No matter which one it is, it divids $(x-3)^2(x-5)^2(x-8)^2$. Then $(T-3I)^2(T-5I)^2(T-8I)^2 = 0$.

Exercise 8.10.

(1) Give an example of an operator on $\mathbb{C}^4$ whose characteristic polynomial equals $(x-1)(x-3)^3$ and whose minimal polynomial equals $(x-1)(x-3)^2$.

Solution 8.10. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

(2) Give an example of an operator on $\mathbb{C}^4$ whose characteristic and minimal polynomials both equal $x(x-1)^2(x-3)$.

Solution 8.10. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

Exercise 8.11. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if the minimal polynomial of T has no repeated zeros.

Solution 8.11. The minimal polynomial of T has no repeated zeros means that the biggest Jordan block is 1×1.

($\Rightarrow$): If V has a basis consisting of eigenvectors of T, then T is diagonalizable. Then the biggest Jordan block is 1×1. Then the minimal polynomial of T has no repeated zeros.

($\Leftarrow$): If the minimal polynomial of T has no repeated zeros, then the biggest Jordan block is 1×1. Then T is diagonalizable. Therefore V has a basis consisting of eigenvectors of T.

Exercise 8.12. Let $N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Please find the characteristic polynomial and minimal polynomial of N.

Solution 8.12. The characteristic polynomial is

$$\det(N - xI) = \det\left(\begin{bmatrix} -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & 1 \\ 0 & 0 & 0 & -x \end{bmatrix}\right) = x^4.$$

Since this is already a Jordan block of size 4 with eigenvalue 0, the minimal polynomial is x^4.

Exercise 8.13. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. Please find its Jordan canonical form C and find the transformation matrix P such that $C = P^{-1}AP$.

Solution 8.13. First find the characteristic polynomial:

$$\det\left(A - xI\right) = \det\left(\begin{bmatrix} 1-x & 2 & 3 \\ 0 & 1-x & 2 \\ 0 & 0 & 2-x \end{bmatrix}\right) = (1-x)^2(2-x).$$

Then the matrix has eigenvalue 1 with multiplicity 2 and eigenvalue 2 with multiplicity 1.

Then try $(I - A)(2I - A)$:

$$(I - A)(2I - A) = \begin{bmatrix} 0 & -2 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0.$$

Therefore the minimal polynomial is not $(1-x)(2-x)$. So the minimal polynomial has to be the characteristic polynomial. Then the Jordan canonical form of the matrix A is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

To find the transformation matrix P, we need to find a basis consisting of eigenvectors of eigenvalue 1 and 2 and a generalized eigenvector of eigenvalue 1.

Eigenvalue 1: Solve $(A-I)X=0$:

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} X = 0.$$

The solution is $X \in \texttt{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$. So the first basis vector can be $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Generalized eigenvector of eigenvalue 1: Solve $(A-I)X = v_1$:

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The solution is $X \in \texttt{Span}\left(\begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}\right)$. We can choose the second basis vector to be $v_2 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$.

Eigenvalue 2: Solve $(A-2I)X=0$:

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} X = 0.$$

The solution is $X \in \texttt{Span}\left(\begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}\right)$. We can choose the third basis vector to be $v_3 = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$.

Therefore $\mathcal{B} = \{v_1, v_2, v_3\}$ is a Jordan basis. The transformation matrix is

$$P = \begin{bmatrix} 1 & 0 & 7 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We may jusitify our answer by computing

$$P^{-1}AP = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Exercise 8.14.

(1) $\mathtt{im}(T^k) \supset \mathtt{im}(T^{k+1})$ for any $k \geq 0$.

(2) If $\mathtt{im}(T^m) = \mathtt{im}(T^{m+1})$, then $\mathtt{im}(T^{m+k}) = \mathtt{im}(T^{m+k+1})$ for any $k \geq 0$.

(3) Let $\dim V = n$. Then $\mathtt{im}(T^n) = \mathtt{im}(T^{n+1})$.

Solution 8.14.

(1) For any $v \in \mathtt{im}(T^{k+1})$, there exists $w \in V$ such that $v = T^{k+1}(w)$. Then $v = T^k(t(w))$. So $v \in \mathtt{im}(T^k)$. So $\mathtt{im}(T^k) \supset \mathtt{im}(T^{k+1})$.

(2) Assume that $\mathtt{im}(T^m)\,\mathtt{im}(T^{m+1})$. For any $v \in \mathtt{im}(T^{m+k})$, there exists $w \in V$ such that $v = T^{m+k}(w) = T^k(T^m(w))$. Then since $T^m(w) \in \mathtt{im}(T^m) = \mathtt{im}(T^{m+1})$, there exists $z \in V$ such that $T^m(w) = T^{m+1}(z)$. Then $v = T^k(T^{m+1}(z)) = T^{m+k+1}(z)$. So $v \in \mathtt{im}(T^{m+k+1})$. Then $\mathtt{im}(T^{m+k}) \subset \mathtt{im}(T^{m+k+1})$. By (1), $\mathtt{im}(T^{m+k}) \supset \mathtt{im}(T^{m+k+1})$. Then $\mathtt{im}(T^{m+k}) = \mathtt{im}(T^{m+k+1})$.

(3) Assume that $\mathtt{im}(T^n) \neq \mathtt{im}(T^{n+1})$. Then by (2), there is a decreasing chain

$$V \supsetneq \mathtt{im}(T) \supsetneq \mathtt{im}(T^2) \supsetneq \ldots \supsetneq \mathtt{im}(T^n) \supsetneq \mathtt{im}(T^{n+1}).$$

Then we have $n = \dim V > \dim \mathtt{im}(T) > \ldots > \dim \mathtt{im}(T^n) > \dim \mathtt{im}(T^{n+1})$. Then $\dim \mathtt{im}(T^{n+1})$ has to be ≤ -1, which is impossible. Then $\mathtt{im}(T^n) = \mathtt{im}(T^{n+1})$.