MATH 132 discussion
University of California, Riverside

Ryan Ta
Spring 2019

MATH 132 Homework 7

6.1. Let V be an inner product space. Prove the following:

(1) 0 is orthogonal to every vector in V.

Proof. Let $v \in V$ be an arbitrary vector. Then $\langle 0, v\rangle = 0$, and so 0 is orthogonal to v. □

(2) 0 is the only vector in V that is orthogonal to itself.

Proof. Let $v \in V$ be a vector that is orthogonal to itself. Then we have $\langle v, v\rangle = 0$. By the definiteness property of the inner product for V, we must have $v = 0$. So 0 is the only vector in V that is orthogonal to itself. □

6.2. Suppose V is a real inner product space.

(1) Show that $\langle u + v, u - v\rangle = \|u\|^2 - \|v\|^2$ for any $u, v \in V$.

Proof. Since V is a real inner product space, conjugate symmetry is the same as symmetry for the inner product; that is, $\langle u, v\rangle = \overline{\langle v, u\rangle} = \langle v, u\rangle$. Therefore, for all $u, v \in V$, we have

$$\begin{aligned}\langle u + v, u - v\rangle &= \langle u, u\rangle - \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle \\ &= \|u\|^2 - \langle u, v\rangle + \langle u, v\rangle - \|v\|^2 \\ &= \|u\|^2 - \|v\|^2,\end{aligned}$$

as desired. □

(2) Show that if $\|u\| = \|v\|$, then $u + v$ is orthogonal to $u - v$.

Proof. Using part (1) and the assumption of part (2), we have

$$\begin{aligned}\langle u + v, u - v\rangle &= \|u\|^2 - \|v\|^2 \\ &= \|v\|^2 - \|v\|^2 \\ &= 0,\end{aligned}$$

and so $u + v$ is orthogonal to $u - v$. □

6.3. Prove or disprove: There is an inner product on $\mathbb{R}^2$ such that the associated norm is given by

$$\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| = \max\{x, y\}$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Proof. Disprove. Consider the nonzero vector $\begin{bmatrix} -1 \\ 0 \end{bmatrix} \in \mathbb{R}^2$. Then

$$\begin{aligned}\left\langle \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\rangle &= \left\|\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\|^2 \\ &= (\max\{-1, 0\})^2 \\ &= 0^2 \\ &= 0,\end{aligned}$$

which violates the definiteness of the inner product. □

6.4. Let V be an inner product space with the inner product $\langle \cdot, \cdot\rangle$. Suppose $S \in \mathcal{L}(V)$ is an injective operator on V. Define a new pairing $\langle \cdot, \cdot\rangle_S$ by $\langle u, v\rangle_S = \langle Su, Sv\rangle$ for all $u, v \in V$.

(1) Please show that $\langle \cdot, \cdot\rangle_S$ is an inner product on V.

Proof. *Positivity:* For all $u \in V$, we have

$$\begin{aligned}\langle u, u\rangle_S &= \langle Su, Su\rangle \\ &= \|Su\|^2 \\ &\geq 0.\end{aligned}$$

Definiteness: From $\langle u, u\rangle_S = \|Su\|^2$, we have $\langle u, u\rangle_S = 0$ if and only if $\|Su\|^2 = 0$. Since $\langle \cdot, \cdot\rangle$ is an inner product for V, we have $\|Su\|^2 = 0$ if and only if $Su = 0$. Finally, since S is injective, it has trivial kernel; that is, $Su = 0$ if and only if $u = 0$.

Additivity in the first slot: For all $u, v, w \in V$, we have

$$\begin{aligned}\langle u + v, w\rangle_S &= \langle S(u + v), Sw\rangle \\ &= \langle Su + Sv, Sw\rangle \\ &= \langle Su, Sw\rangle + \langle Sv, Sw\rangle \\ &= \langle u, w\rangle_S + \langle v, w\rangle_S.\end{aligned}$$

Homogeneity in the first slot: For all $u, v \in V$ and $\lambda \in \mathbb{F}$, we have

$$\begin{aligned}\langle \lambda u, v\rangle_S &= \langle S(\lambda u), Sv\rangle \\ &= \langle \lambda Su, Sv\rangle \\ &= \lambda\langle Su, Sv\rangle \\ &= \lambda\langle u, v\rangle_S.\end{aligned}$$

Conjugate symmetry: For all $u, v \in V$, we have

$$\begin{aligned}\langle u, v\rangle_S &= \langle Su, Sv\rangle \\ &= \overline{\langle Sv, Su\rangle} \\ &= \overline{\langle v, u\rangle_S}.\end{aligned}$$

These properties verify that $\langle \cdot, \cdot\rangle_S$ defines an inner product on V. □

(2) Please give a counterexample that $\langle \cdot, \cdot\rangle_S$ is not an inner product when S is not injective.

Proof. Let $V := \mathbb{R}^2$, let $\langle \cdot, \cdot\rangle$ be the Euclidean dot product, and let $v := \begin{bmatrix}1\\1\end{bmatrix} \in \mathbb{R}^2$. Since S is not injective, we can for instance let S be the zero map. So we have

$$\begin{aligned}\langle v, v\rangle_S &= \left\langle \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\rangle_S \\ &= \left\langle S\left(\begin{bmatrix}1\\1\end{bmatrix}\right), S\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\right\rangle \\ &= \langle 0, 0\rangle \\ &= 0.\end{aligned}$$

This violates the definiteness property of the inner product, and so $\langle \cdot, \cdot\rangle_S$ is not an inner product. □

6.5. Let $\mathbb{R}^3$ be the inner product space with the usual dot product. Let $T \in \mathcal{L}(\mathbb{R}^3)$ have an upper-triangular matrix with respect to the basis

$$\left\{ w_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, w_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}, w_3 = \begin{bmatrix}1\\1\\2\end{bmatrix} \right\}.$$

Find an orthonormal basis of $\mathbb{R}^3$ with respect to which T has an upper-triangular matrix.

Proof. Based on Theorem 6.2.13 and its proof in the Notes, we will use the Gram-Schmidt procedure to find an orthonormal basis of $\{w_1, w_2, w_3\}$ such that T is upper-triangular with respect to said orthonormal basis. Following this procedure for the first basis vector w_1, we obtain

$$\begin{aligned} e_1 &= \frac{w_1}{\|w_1\|} \\ &= \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\0\\0\end{bmatrix}\right\|} \\ &= \begin{bmatrix}1\\0\\0\end{bmatrix}. \end{aligned}$$

Doing the same procedure for w_1, w_2, we obtain

$$\begin{aligned} e_2 &= \frac{w_2 - \langle w_2, e_1\rangle e_1}{\|w_2 - \langle w_2, e_1\rangle e_1\|} \\ &= \frac{\begin{bmatrix}1\\1\\1\end{bmatrix} - \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\rangle \begin{bmatrix}1\\0\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\\1\end{bmatrix} - \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\rangle \begin{bmatrix}1\\0\\0\end{bmatrix}\right\|} \\ &= \frac{\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\right\|} \\ &= \frac{\begin{bmatrix}0\\1\\1\end{bmatrix}}{\left\|\begin{bmatrix}0\\1\\1\end{bmatrix}\right\|} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix} \\ &= \begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}. \end{aligned}$$

Finally, doing the same procedure for w_1, w_2, w_3, we obtain

$$
\begin{aligned}
e_3 &= \frac{w_3 - \langle w_3, e_1\rangle e_1 - \langle w_3, e_2\rangle e_2}{\|w_3 - \langle w_3, e_1\rangle e_1 - \langle w_3, e_2\rangle e_2\|} \\
&= \frac{\begin{bmatrix}1\\1\\2\end{bmatrix} - \left\langle \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\rangle \begin{bmatrix}1\\0\\0\end{bmatrix} - \left\langle \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\right\rangle \begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}}{\left\| \begin{bmatrix}1\\1\\2\end{bmatrix} - \left\langle \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\rangle \begin{bmatrix}1\\0\\0\end{bmatrix} - \left\langle \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\right\rangle \begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\right\|} \\
&= \frac{\begin{bmatrix}1\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{3}{\sqrt{2}}\begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}}{\left\| \begin{bmatrix}1\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{3}{\sqrt{2}}\begin{bmatrix}0\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\right\|} \\
&= \frac{\begin{bmatrix}0\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}}{\left\|\begin{bmatrix}0\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\right\|} \\
&= \sqrt{2}\begin{bmatrix}0\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix} \\
&= \begin{bmatrix}0\\-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}.
\end{aligned}
$$

So $\{e_1, e_2, e_3\} \subset \mathbb{R}^3$ is an orthonormal basis for which T has an upper-triangular matrix. □