Solutions to Exercises 6

Exercise 6.1. Let V be an inner product space.

(1) 0 is orthogonal to every vector in V.

(2) 0 is the only vector in V that is orthogonal to itself.

Solution 6.1.

(1) For any $u \in V$, $\langle 0, u\rangle = 0$. So 0 is orthogonal to u.

(2) If $\langle u, u\rangle = 0$, then $u = 0$. So 0 is the only vector in V that is orthogonal to itself.

Exercise 6.2. Suppose V is a real inner product space.

(1) Show that $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$ for any $u, v \in V$.

(2) Show that if $\|u\| = \|v\|$, then $u + v$ is orthogonal to $u - v$.

Solution 6.2.

(1) For any $u, v \in V$, $\langle u+v, u-v\rangle = \langle u, u\rangle - \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle = \|u\|^2 - \|v\|^2$.

(2) If $\|u\| = \|v\|$, since $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2 = 0$, $u + v$ is orthogonal to $u - v$.

Exercise 6.3. Prove or disprove: there is an inner product on R^2 such that the associated norm is given by $\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \max\{x, y\}$ for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Solution 6.3. If there is, then $\|v\| = 0$ if and only if $v = 0$. However $\left\| \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\| = 0$ while $\begin{bmatrix} 0 \\ -1 \end{bmatrix} \neq 0$. So such an inner product doesn't exist.

Exercise 6.4. Let V be an inner product space with the inner product $\langle \cdot, \cdot\rangle$. Suppose $S \in \mathcal{L}(V)$ is an injective operator on V. Define a new pairing $\langle \cdot, \cdot\rangle_S$ by $\langle u, v\rangle_S = \langle Su, Sv\rangle$ for $u, v \in V$.

(1) Please show that $\langle \cdot, \cdot\rangle_S$ is an inner product on V.

(2) Please give a counter example that $\langle \cdot, \cdot\rangle_S$ is not an inner product when S is not injective.

Solution 6.4.

(1) Let us check the five axioms for $\langle\cdot,\cdot\rangle_S$.

positivity: $\langle v,v\rangle_S = \langle Sv, Sv\rangle \geq 0$ for all $v \in V$.

definiteness: $\langle v,v\rangle_S = 0$ if and only if $\langle Sv, Sv\rangle = 0$ if and only if $Sv = 0$. Since S is injective, then $Sv = 0$ if and only if $v = 0$.

additivity in first slot: $\langle u+v,w\rangle_S = \langle S(u+v), Sw\rangle = \langle Su+Sv, Sw\rangle = \langle Su, Sw\rangle + \langle Sv, Sw\rangle = \langle u,w\rangle_S + \langle v,w\rangle_S$ for all $u,v,w \in V$.

homogeneity in first slot: $\langle \lambda u,v\rangle_S = \langle S(\lambda u), Sv\rangle = \langle \lambda Su, Sv\rangle = \lambda\langle Su, Sv\rangle = \lambda\langle u,v\rangle_S$ for all $u,v \in V$ and $\lambda \in \mathbb{F}$.

conjugate symmetry: $\langle u,v\rangle_S = \langle Su, Sv\rangle = \overline{\langle Sv, Su\rangle} = \overline{\langle v,u\rangle_S}$ for all $u,v \in V$.

Then $\langle\cdot,\cdot\rangle_S$ is an inner product on V.

(2) Consider $\mathbb{R}^2$ and the usual dot product. Let S is defined by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. $\langle\cdot,\cdot\rangle_S$ is not an inner product.

Exercise 6.5. Let $\mathbb{R}^3$ be the inner product space with the usual dot product. Let $T \in \mathcal{L}(\mathbb{R}^3)$ has an upper-triangular matrix with respect to the basis $\left\{ w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$. Find an orthonormal basis of $\mathbb{R}^3$ with respect to which T has an upper-triangular matrix.

Solution 6.5. By the proof of Theorem 6.2.13, what we need to do is to apply the Gram-Schmidt procedure to the basis.

(1) $e_1 = w_1/\|w_1\| = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(2) $v_2 = w_2 - \langle w_2, e_1\rangle e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then $e_2 = v_2/\|v_2\| = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

(3) $v_3 = w_3 - \langle w_3, e_1\rangle\, e_1 - \langle w_3, e_2\rangle\, e_2 = \begin{bmatrix}1\\1\\2\end{bmatrix} - \left\langle \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\rangle \begin{bmatrix}1\\0\\0\end{bmatrix} - \left\langle \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \right\rangle \begin{bmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} =$

$\begin{bmatrix}0\\-1/2\\1/2\end{bmatrix}$. Then $e_3 = v_3/\|v_3\| = \begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$.

Then $\{e_1, e_2, e_3\}$ is an orthonormal basis such that the matrix of T with respect to this basis is upper-triangular.