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MATH 132 Homework 8

7.1. Consider $\mathbb{C}^n$ with the dot product. Define $T \in \mathcal{L}(\mathbb{C}^n)$ by

$$T\left(\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}\right) = \begin{bmatrix} 0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix}.$$

Find a formula for $T^*\left(\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}\right)$.

Proof. Let $\mathcal{E} := \{e_1, \dots, e_n\} \subset \mathbb{C}^n$ be the standard basis, which is by definition orthonormal. Since T sends z_1 to 0 and z_i to z_{i-1} for all $i = 2, \dots, n$, and since we have

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

where the 1 is located in the i^{th} position of the vector, we get

$$T(e_i) = \begin{cases} e_{i+1} & \text{if } i = 1, \dots, n-1 \\ 0 & \text{if } i = n. \end{cases}$$

Next, since $\mathcal{E} \subset \mathbb{C}^n$ is a basis, we can write the adjoint of T as

$$T^*(e_j) = a_{1j}e_1 + \cdots + a_{nj}e_n$$

for some choice of scalars $a_{ij} \in \mathbb{C}$ for $i, j = 1, \dots, n$. Also we recall the Kronecker delta

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Since $\mathcal{E}$ is orthonormal, we can find the complex conjugate of the scalars as follows:

$$\begin{aligned} \overline{a_{ij}} &= \overline{a_{1j}}0 + \cdots + \overline{a_{ij}}1 + \cdots + \overline{a_{nj}}0 \\ &= \overline{a_{1j}}\langle e_i, e_1 \rangle + \cdots + \overline{a_{ij}}\langle e_i, e_i \rangle + \cdots + \overline{a_{nj}}\langle e_i, e_n \rangle \\ &= \langle e_i, a_{1j}e_1 \rangle + \cdots + \langle e_i, a_{ij}e_1 \rangle + \cdots + \langle e_i, a_{nj}e_1 \rangle \\ &= \langle e_i, a_{1j}e_1 + \cdots + a_{nj}e_n \rangle \\ &= \langle e_i, T^*(e_j) \rangle \\ &= \langle T(e_i), e_j \rangle \\ &= \begin{cases} \langle e_{i+1}, e_j \rangle & \text{if } i = 1, \dots, n-1 \\ \langle 0, e_j \rangle & \text{if } i = n \end{cases} \\ &= \begin{cases} \delta_{i+1,j} & \text{if } i = 1, \dots, n-1 \\ 0 & \text{if } i = n \end{cases} \\ &= \begin{cases} 1 & \text{if } i = j-1 \text{ for all } j = 2, \dots, n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Our final expression of $\overline{a_{ij}}$ implies that we have $a_{ij} \in \mathbb{R} \subset \mathbb{C}$, depending on our choices of $i, j = 1, \dots, n$; this means in particular $\overline{a_{ij}} = a_{ij}$. Therefore, for $j = 1$, we have

$$\begin{aligned} T^*(e_1) &= a_{11}e_1 + \cdots + a_{n1}e_n \\ &= 0e_1 + \cdots + 0e_n \\ &= 0, \end{aligned}$$

and, for all $j = 2, \dots, n$, we have

$$\begin{aligned} T^*(e_j) &= a_{1j}e_1 + \cdots + a_{nj}e_n \\ &= \overline{a_{1j}}e_1 + \cdots + \overline{a_{nj}}e_n \\ &= \overline{a_{1j}}e_1 + \cdots + \overline{a_{j-1,j}}e_{j-1} + \cdots + \overline{a_{nj}}e_n \\ &= 0e_1 + \cdots + 1e_{j-1} + \cdots + 0e_n \\ &= e_{j-1}. \end{aligned}$$

In summary, for all $j = 1, \dots, n$, we have

$$T^*(e_j) = \begin{cases} 0 & \text{if } j = 1 \\ e_{j-1} & \text{if } j = 2, \dots, n, \end{cases}$$

which implies that T^* sends z_i to z_{i+1} for all $i = 1, \dots, n-1$ and z_n to 0. In other words,

$$T^*\left(\begin{bmatrix} z_1 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix}\right) = \begin{bmatrix} z_2 \\ \vdots \\ z_n \\ 0 \end{bmatrix}$$

is the formula for T^*. □

7.2. Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

Proof. Since T satisfies $T^9 = T^8$, or $T^9 - T^8 = 0$, it follows that $x^8(x-1)$ is a polynomial multiple of the minimal polynomial of T. This also implies that at least one of 0 and 1 are the only possible eigenvalues of T (up to their multiplicities). In other words, all the eigenvalues of T are real numbers, which means we conclude by Corollary 7.2.4 of the Notes that T is self-adjoint.

Furthermore, since T is normal, the Complex Spectral Theorem (Theorem 7.2.1 of the Notes) asserts that T has a diagonal matrix with respect to some orthonormal basis of V. Since the matrix of T is diagonal, it follows that at least one of, if not both, the Jordan blocks J_0, J_1 is/are of size 1×1, meaning that the minimal polynomial of T either is equal to or divides $x(x-1)$. In either case, we have $T(T-I) = 0$, or $T^2 = T$. □

7.3. Let $S, T \in \mathcal{L}(V)$ be self-adjoint. Show that ST is self-adjoint if and only if $ST = TS$.

Proof. We will prove the forward direction: If ST is self-adjoint, then $ST = TS$. Since S, T are self-adjoint by the hypotheses, we have $S = S^*$ and $T = T^*$. Since ST is self-adjoint by assumption, we have $ST = (ST)^*$. Finally, by Proposition 7.1.4, part (5), of the Notes, we have $(ST)^* = T^*S^*$. Therefore, we have

$$\begin{aligned} ST &= (ST)^* \\ &= T^*S^* \\ &= TS, \end{aligned}$$

as desired. Now we will prove the backward direction: If $ST = TS$, then ST is self-adjoint. Since again we have $S = S^*$ and $T = T^*$, and now we are assuming $ST = TS$, we have

$$\begin{aligned} (ST)^* &= T^*S^* \\ &= TS \\ &= ST, \end{aligned}$$

as desired. □

7.4. Suppose V is a complex inner product space with $V \neq \{0\}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Proof. Let $V = \mathbb{C}^2$, let $i \in \mathbb{C}$ be a scalar, and let T be the identity map (that is, $Tv = v$ for all $v \in \mathbb{C}^2$). Then T is self-adjoint; that is, $T^* = T$. But we also have $\bar{i}T^* = -iT \neq iT^*$, which signifies that the set of self-adjoint operators on V is not closed under scalar multiplication. So the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$. □

7.5. Give an example of an operator T on a complex vector space such that $T^9 = T^8$ but $T^2 \neq T$.

Proof. Let $\mathbb{C}^4$ be the complex vector space, and let $T \in \mathcal{L}(\mathbb{C}^4)$ be defined by its matrix representation A with respect to the standard basis of $\mathbb{C}^4$:

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $T^9 = T^8$ because

$$A^9 = A^8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

but

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \neq A,$$

as desired. □

Explanation for why we chose A as above: We notice that A is a matrix in Jordan canonical form. Since we have $T \in \mathcal{L}(\mathbb{C}^4)$, the matrix A must have size 4×4, and so the characteristic polynomial of A must have degree 4. Since A also satisfies $A^9 = A^8$, the characteristic polynomial of A divides $x^8(x-1)$, and so $0, 1$ are the only eigenvalues of A. Since A also satisfies $A^2 \neq A$, the minimal polynomial of A cannot be $x(x-1)$; the minimal polynomial of A must have at least degree 3 and consist of polynomial multiples of the linear factors $x, x-1$. Finally, since the minimal polynomial divides the characteristic polynomial, the minimal polynomial of A must be one of $x^2(x-1), x(x-1)^2, x^2(x-1)^2, x^3(x-1), x(x-1)^3$, which correspond respectively to possible Jordan canonical forms (unique up to permutation of Jordan blocks corresponding to eigenvalues 0, 1 with their possible multiplicities)

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then A_2, A_3, A_5 do not satisfy our requirements because A_2^m, A_3^m, A_5^m will have entries that depend on m; in particular, $A_2^9 \neq A_2^8, A_3^9 \neq A_3^8, A_5^9 \neq A_5^8$. However, A_1, A_4 both work as they satisfy $A_1^2 \neq A_2^2, A_4^2 \neq A_4^2$ and $A_1^9 = A_1^8, A_4^9 = A_4^8$. So we chose $A = A_4$ (or we can equally choose $A = A_1$) for our matrix in the beginning of this solution.

7.6. Let V be a finite-dimensional complex vector space. Suppose that $T \in \mathcal{L}(V)$ is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

Proof. Since T is a normal operator on V, the Complex Spectral Theorem (Theorem 7.2.1 of the Notes) asserts that there exists a basis of V consisting of orthonormal eigenvectors as the basis vectors. In particular, there exist basis vectors $v_1, v_2 \in V$ that correspond respectively to the eigenvalues 3 and 4 and both have unit length (that is, $\|v_1\| = \|v_2\| = 1$); in other words, $v_1, v_2 \in V$ satisfy $Tv_1 = 3v_1$ and $Tv_2 = 4v_2$. Let $v := v_1 + v_2 \in V$. Then the image of v under T is

$$\begin{aligned} Tv &= T(v_1 + v_2) \\ &= Tv_1 + Tv_2 \\ &= 3v_1 + 4v_2. \end{aligned}$$

Using the Pythagorean Theorem, we obtain

$$\begin{aligned} \|v\|^2 &= \|v_1\|^2 + \|v_2\|^2 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} \|Tv\|^2 &= \|3v_1\|^2 + \|4v_2\|^2 \\ &= 9\|v_1\|^2 + 16\|v_2\|^2 \\ &= 9 + 16 \\ &= 25, \end{aligned}$$

and so we get $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$. □

7.7. Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^4)$ such that T is normal but not self-adjoint.

Proof. Let T be represented by the matrix

$$A := \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with respect to the standard basis of $\mathbb{C}^4$. Then the adjoint (conjugate transpose) of A is

$$\begin{aligned} A^* &= A^H \\ &= \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\neq A, \end{aligned}$$

which means A is not self-adjoint. But we also have

$$\begin{aligned} A^*A &= \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= AA^*, \end{aligned}$$

which means T is normal. □

7.8. Let $T \in \mathcal{L}(V, W)$. Please prove that:

(1) T is injective if and only if T^* is surjective.

Proof. First we will prove the forward direction: If T is injective, then T^* is surjective. We will do a proof by conradiction; suppose to the contrary that T^* is not surjective. Then we have $\operatorname{im} T^* \subsetneq V$, and so, after using the Gram-Schmidt procedure if necessary, there exists an orthonormal basis of $\operatorname{im} T^*$ that can be extended to a basis of V. In particular, there exists at least one basis vector in V but not in $\operatorname{im} T^*$; let v be such a vector. Since the basis vectors in $\operatorname{im} T^*$ form an orthornomal basis of $\operatorname{im} T^*$, it follows that v is orthogonal to all the basis vectors of that orthornomal basis of $\operatorname{im} T^*$ (so that no component of v is in $\operatorname{im} T^*$), which also means that v is orthogonal to all vectors in $\operatorname{im} T^*$; that is, for all $w \in W$ we have $\langle v, T^*w \rangle = 0$. Therefore, for all $w \in W$, we have

$$\begin{aligned} \langle Tv, w \rangle &= \langle v, T^*w \rangle \\ &= 0, \end{aligned}$$

which implies $Tv = 0$. Since $T \in \mathcal{L}(V, W)$ is injective, we get $v = 0$, which is a contradiction because we said earlier that v is also a basis vector; a zero vector is never a basis vector.

Now we will prove the backward direction: If T^* is surjective, then T is injective. Let $v \in V$ satisfy $Tv = 0$. Since $T^* \in \mathcal{L}(W, V)$ is surjective, by definition for any $u \in V$ there exists $w \in W$ such that $T^*w = v$. Therefore, for all $u \in V$, we have

$$\begin{aligned} \langle v, u \rangle &= \langle v, T^*w \rangle \\ &= \langle Tv, w \rangle \\ &= \langle 0, w \rangle \\ &= 0, \end{aligned}$$

which implies $v = 0$. Hence, T is injective. □

Alternate proof using the Axler textbook: By 7.7 of Axler (page 207), we have $\operatorname{im} T^* = (\operatorname{nul} T)^\perp$, which implies in particular $\operatorname{nul} T = \{0\}$ if and only if $\operatorname{im} T = \{0\}^\perp = V$. Hence, T is injective if and only if $\operatorname{nul} T = 0$, if and only if $\operatorname{im} T^* = V$, if and only if T^* is surjective.

(2) T is surjective if and only if T^* is injective.

Proof. Replace T with T^* in the statement of part (1); the statement would read: "T^* is injective if and only if T^{**} is surjective." But we also recall that we have $T^{**} = T$. So the statement really reads: "T^* is injective if and only if T is surjective," which is precisely the statement of part (2). □

Alternate proof using the Axler textbook: By 7.7 of Axler (page 207), we have $\operatorname{nul} T^* = (\operatorname{im} T)^\perp$, which implies in particular $\operatorname{im} T = V$ if and only if $\operatorname{nul} T^* = V^\perp = \{0\}$. Hence, T is surjective if and only if $\operatorname{im} T = V$, if and only if $\operatorname{nul} T^* = \{0\}$, if and only if T^* is injective.

7.9. Consider $\mathbb{C}^3$ with the dot product. Let $\mathcal{E}$ be the standard basis. Let $T \in \mathcal{L}(\mathbb{C}^3)$ be defined by

$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Please find an orthonormal basis such that the matrix of T is diagonal, or prove that such a basis does not exist.

Proof. The standard basis $\mathcal{E}$ is an orthonormal basis, which implies that the adjoint of $[T]_{\mathcal{E}\leftarrow\mathcal{E}}$ is the conjugate transpose of $[T]_{\mathcal{E}\leftarrow\mathcal{E}}$; that is,

$$\begin{aligned}
[T]^*_{\mathcal{E}\leftarrow\mathcal{E}} &= [T]^H_{\mathcal{E}\leftarrow\mathcal{E}} \\
&= \begin{bmatrix} \bar{1} & \bar{0} & \bar{1} \\ \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{1} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.
\end{aligned}$$

Notice that we have

$$\begin{aligned}
TT^* &= [T]_{\mathcal{E}\leftarrow\mathcal{E}} [T]^*_{\mathcal{E}\leftarrow\mathcal{E}} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\
&= [T]^*_{\mathcal{E}\leftarrow\mathcal{E}} [T]_{\mathcal{E}\leftarrow\mathcal{E}} \\
&= T^*T,
\end{aligned}$$

which means T is normal. So the Complex Spectral Theorem (Theorem 7.2.1 of the Notes) asserts that T has a diagonal matrix with respect to some orthonormal basis of eigenvectors in $\mathbb{C}^3$; in other words, $[T]_{\mathcal{E}\leftarrow\mathcal{E}}$ is diagonalizable with respect to the orthonormal basis of eigenvectors. This motivates us to find the eigenvalues. From the equation

$$\begin{aligned}
0 &= \det([T]_{\mathcal{E}\leftarrow\mathcal{E}} - \lambda I) \\
&= \det\left(\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right) \\
&= (1-\lambda)^3 + 1 \\
&= -\lambda^3 + 3\lambda^2 - 3\lambda + 2 \\
&= -(\lambda - 2)(\lambda^2 - \lambda + 1),
\end{aligned}$$

from which we obtain the eigenvalues $\lambda = 2, \frac{1-\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2}$. For $\lambda = 2$, the equation

$$\begin{aligned}
0 &= (A - 2I)x \\
&= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}$$

gives us $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$, and so we can choose

$$\begin{aligned} v_1 &= \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\left\|\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\|} \\ &= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \end{aligned}$$

so that v_1 is a normal vector (that is, $\|v_1\| = 1$). For $\lambda = \frac{1-\sqrt{3}i}{2}$, the equation

$$\begin{aligned} 0 &= \left(A - \frac{1-\sqrt{3}i}{2}I\right)x \\ &= \begin{bmatrix} \frac{1+\sqrt{3}i}{2} & 1 & 0 \\ 0 & \frac{1+\sqrt{3}i}{2} & 1 \\ 1 & 0 & \frac{1+\sqrt{3}i}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

to obtain $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{-1-\sqrt{3}i}{2}x_1 \\ \frac{1-\sqrt{3}i}{2}x_1 \end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ \frac{-1-\sqrt{3}i}{2} \\ \frac{1-\sqrt{3}i}{2} \end{bmatrix}\right\}$, and so we can choose

$$\begin{aligned} v_2 &= \frac{\begin{bmatrix} 1 \\ \frac{-1-\sqrt{3}i}{2} \\ \frac{1-\sqrt{3}i}{2} \end{bmatrix}}{\left\|\begin{bmatrix} 1 \\ \frac{-1-\sqrt{3}i}{2} \\ \frac{1-\sqrt{3}i}{2} \end{bmatrix}\right\|} \\ &= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ \frac{-1-\sqrt{3}i}{2} \\ \frac{1-\sqrt{3}i}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \frac{-1-\sqrt{3}i}{2\sqrt{3}} \\ \frac{1-\sqrt{3}i}{2\sqrt{3}} \end{bmatrix}, \end{aligned}$$

so that v_2 is a normal vector (that is, $\|v_2\| = 1$). For $\lambda = \frac{1+\sqrt{3}i}{2}$, the equation

$$\begin{aligned} 0 &= \left(A - \frac{1+\sqrt{3}i}{2}I\right)x \\ &= \begin{bmatrix} \frac{1-\sqrt{3}i}{2} & 1 & 0 \\ 0 & \frac{1-\sqrt{3}i}{2} & 1 \\ 1 & 0 & \frac{1-\sqrt{3}i}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

to obtain $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{-1+\sqrt{3}i}{2} x_1 \\ \frac{-1-\sqrt{3}i}{2} x_1 \end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix} 1 \\ \frac{-1+\sqrt{3}i}{2} \\ \frac{-1-\sqrt{3}i}{2} \end{bmatrix} \right\}$, and so we can choose

$$v_3 = \frac{\begin{bmatrix} 1 \\ \frac{-1-\sqrt{3}i}{2} \\ \frac{1-\sqrt{3}i}{2} \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ \frac{-1-\sqrt{3}i}{2} \\ \frac{1-\sqrt{3}i}{2} \end{bmatrix} \right\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \frac{-1+\sqrt{3}i}{2} \\ \frac{-1-\sqrt{3}i}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-1+\sqrt{3}i}{2\sqrt{3}} \\ \frac{-1-\sqrt{3}i}{2\sqrt{3}} \end{bmatrix},$$

so that v_3 is a normal vector (that is, $\|v_3\| = 1$). So we have a basis of orthonormal eigenvectors $\mathcal{B} := \{v_1, v_2, v_3\}$; our change-of-basis matrix is

$$P_{\mathcal{B}\leftarrow\mathcal{E}} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 1 & 1 \\ \frac{1}{\sqrt{3}} & \frac{-1-\sqrt{3}i}{2\sqrt{3}} & \frac{-1+\sqrt{3}i}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1-\sqrt{3}i}{2\sqrt{3}} & \frac{-1-\sqrt{3}i}{2\sqrt{3}}. \end{bmatrix}$$

The matrix of T with respect to $\mathcal{B}$ is

$$[T]_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1-\sqrt{3}i}{2} & 0 \\ 0 & 0 & \frac{1+\sqrt{3}i}{2} \end{bmatrix},$$

which is a diagonal matrix of eigenvalues that satisfies

$$P_{\mathcal{E}\leftarrow\mathcal{B}} [T]_{\mathcal{B}\leftarrow\mathcal{B}} P^{-1}_{\mathcal{E}\leftarrow\mathcal{B}} = [T]_{\mathcal{E}\leftarrow\mathcal{E}}.$$

(The computation itself with these matrices to verify this equation is rather tedious and therefore not required to be done.) □