Solutions to Exercises 7

Exercise 7.1. Consider $\mathbb{C}^n$ with the dot product. Define $T \in \mathbb{C}^n$ by

$$T\left(\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}\right) = \begin{bmatrix} 0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix}.$$

Find a formula for $T^*\left(\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}\right)$. You cannot directly use Theorem 7.1.6.

Solution 7.1. Choose the standard basis $\mathcal{E} = \{e_1, \ldots, e_n\}$. Then $\mathcal{E}$ is an orthonormal basis. From the definition of T, we have that $T(e_i) = e_{i+1}$ for $i = 1, \ldots, n-1$ and $T(e_n) = 0$.

Let $T^*(e_j) = a_{1j}e_1 + \ldots + a_{nj}e_n$. To find a_{ij}, we need to compute $\langle e_i, a_{1j}e_1 + \ldots + a_{nj}e_n \rangle = \overline{a_{ij}}$. Then

$$\begin{aligned} \overline{a_{ij}} &= \langle e_i, a_{1j}e_1 + \ldots + a_{nj}e_n \rangle = \langle e_i, T^*(e_j) \rangle = \langle T(e_i), e_j \rangle \\ &= \begin{cases} \langle e_{i+1}, e_j \rangle & i = 1, \ldots, n-1, \\ \langle 0, e_j \rangle & i = n, \end{cases} \\ &= \begin{cases} \delta_{i+1,j} & i = 1, \ldots, n-1, \\ 0 & i = n, \end{cases} \end{aligned}$$

Then this means that $a_{j-1,j} = 1$ for $j = 2, \ldots, n$ and $a_{ij} = 0$ otherwise. So $T^*(e_1) = 0$ and $T^*(e_j) = e_{j-1}$ for $j = 2, \ldots, n$. Then

$$T^*\left(\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}\right) = \begin{bmatrix} z_2 \\ \vdots \\ z_n \\ 0 \end{bmatrix}.$$

Exercise 7.2. Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

Solution 7.2. Since $T^9 = T^8$, $x^9 - x^8$ is a multiple of the minimal polynomial of T. Then the eigenvalues of T should be 0 or 1 or both. Then all eigenvalues of T are real. Since T is normal, by Corollary 7.2.4 T is self-adjoint.

Since T is diagonal, the biggest Jordan block is of size 1. Then the minimal polynomial of T should contain at most one factor x and one factor $(x-1)$. So the minimal polynomial divides $x(x-1)$. Then $T(T-I) = 0$. Therefore $T^2 = T$.

Exercise 7.3. Let $S, T \in \mathcal{L}(V)$ be self-adjoint. Show that ST is self-adjoint if and only if $ST = TS$.

Solution 7.3.

(⇒): Since S, T, ST are self-adjoint, then $S^* = S$, $T^* = T$ and $(ST)^* = ST$. Then $ST = (ST)^* = T^*S^* = TS$.

(⇐): Since $S^* = S$, $T^* = T$ and $ST = TS$, we have $(ST)^* = T^*S^* = TS = ST$. So ST is self-adjoint.

Exercise 7.4. Suppose V is a complex inner product space with $V \neq \{0\}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

Solution 7.4. $\overline{\lambda} \neq \lambda$ for most $\lambda \in \mathbb{C}$. Therefore if T is self-adjoint, then $(\lambda T)^* = \overline{\lambda} T^* = \overline{\lambda} T \neq \lambda T$ in general. So self-adjoint operators don't form a subspace of $\mathcal{L}(V)$.

Exercise 7.5. Give an example of an operator T on a complex vector space such that $T^9 = T^8$ but $T^2 \neq T$.

Solution 7.5. From the previous exercise, the biggest difference here is that T doesn't have to be diagonalizable. So to find a counter example we should look at those non-diagonalizable operators. For example, let $T \in \mathbb{C}^4$ be defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$A^9 = A^8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \neq A.$$

Exercise 7.6. Let V be a finite-dimensional complex vector space. Suppose that T is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

Solution 7.6. Since T is normal, there exists an orthonormal basis such that all basis vectors are eigenvectors. Let v_1 be the one with eigenvalue 3 and v_2 be the one with eigenvalue 4. Let $v = v_1 + v_2$. Then

$$\|v\| = \sqrt{\|v\|^2} = \sqrt{\|v_1\|^2 + \|v_2\|^2} = \sqrt{2},$$

and

$$\|Tv\| = \|T(v_1 + v_2)\| = \|3v_1 + 4v_2\| = \sqrt{\|3v_1\|^2 + \|4v_2\|^2} = 5.$$

Exercise 7.7. Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^4)$ such that T is normal but not self-adjoint.

Solution 7.7. There are plenty based on Proposition 7.2.2. Here is one: Let $\mathcal{E}$ be an orthonormal basis. T is defined by $\begin{bmatrix} T \end{bmatrix}_{\mathcal{E}\leftarrow\mathcal{E}} = A = \begin{bmatrix} i & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$. It is not self-adjoint since

$A^H = \begin{bmatrix} -i & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \neq A$. However $AA^H = A^H A$.

Exercise 7.8. Let $T \in \mathcal{L}(V, W)$. Please prove that

(1) T is injective if and only if T^* is surjective.

(2) T is surjective if and only if T^* is injective.

Solution 7.8.

(1) **(⇒):** Assume T is injective. Assume that T^* is not surjective. Then $\mathtt{im}(T^*) \subsetneq V$. Choose a basis of $\mathtt{im}(T^*)$, extend it to be a basis of V, and apply the Gram-Schmidt procedure to get an orthonormal basis. Pick v to be any one basis vector outside $\mathtt{im}(T^*)$. Since the first $\dim \mathtt{im}(T^*)$ basis vectors form an orthonormal basis of $\mathtt{im}(T^*)$, v should be orthogonal to all vectors in $\mathtt{im}(T^*)$. Then for any $w \in W$, $\langle Tv, w\rangle = \langle v, T^*w\rangle = 0$. So $Tv = 0$. Then by T being injective, $v = 0$. This is a contradiction. So $\mathtt{im}(T^*) = V$. Then T^* is surjective.

(⇐): Assume T^* is surjective. Assume $Tv = 0$ for some $v \in V$. Then for any $w \in W$, $\langle v, T^*w\rangle = \langle Tv, w\rangle = 0$. Since T^* is surjective, this means that $\langle v, u\rangle = 0$ for any $u \in V$. Then $v = 0$. So T is injective.

(2) It follows from the first part by $T = (T^*)^*$.

Exercise 7.9. Consider $\mathbb{C}^3$ with the dot product. Let $\mathcal{E}$ be the standard basis. Let $T \in \mathcal{L}(\mathbb{C}^3)$ be defined by

$$\left[T\right]_{\mathcal{E}\leftarrow\mathcal{E}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Please find an orthonormal basis such that the matrix of T is diagonal, or prove that such a basis doesn't exist.

Solution 7.9. Since $\mathcal{E}$ is an orthonormal basis,

$$\left[T^*\right]_{\mathcal{E}\leftarrow\mathcal{E}} = \left[T\right]_{\mathcal{E}\leftarrow\mathcal{E}}^H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

T is normal. Then T can be diagonalized by an orthonormal basis, which are also eigenvectors. Solve

$$\det\left(\left[T\right]_{\mathcal{E}\leftarrow\mathcal{E}} - \lambda I\right) = 0.$$

Then $\lambda = 2$, $\frac{1+\sqrt{3}i}{2}$ and $\frac{1-\sqrt{3}i}{2}$. These are three eigenvalues.

$\lambda = 2$**:** Solve $\begin{bmatrix} 1-2 & 1 & 0 \\ 0 & 1-2 & 1 \\ 1 & 0 & 1-2 \end{bmatrix} X = 0$. The solution is $X \in \texttt{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$. Then we choose $v_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ since we need $\|v_1\| = 1$.

$\lambda = \frac{1+\sqrt{3}i}{2}$: Solve $\begin{bmatrix} 1-\frac{1+\sqrt{3}i}{2} & 1 & 0 \\ 0 & 1-\frac{1+\sqrt{3}i}{2} & 1 \\ 1 & 0 & 1-\frac{1+\sqrt{3}i}{2} \end{bmatrix} X = 0$. The solution is $X \in \mathtt{Span}\left(\begin{bmatrix} \frac{-1+\sqrt{3}i}{2} \\ \frac{-1-\sqrt{3}i}{2} \\ 1 \end{bmatrix}\right)$.

Then we choose $v_2 = \begin{bmatrix} \frac{-1+\sqrt{3}i}{2\sqrt{3}} \\ \frac{-1-\sqrt{3}i}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ since we need $\|v_2\| = 1$.

$\lambda = \frac{1-\sqrt{3}i}{2}$: Solve $\begin{bmatrix} 1-\frac{1-\sqrt{3}i}{2} & 1 & 0 \\ 0 & 1-\frac{1-\sqrt{3}i}{2} & 1 \\ 1 & 0 & 1-\frac{1-\sqrt{3}i}{2} \end{bmatrix} X = 0$. The solution is $X \in \mathtt{Span}\left(\begin{bmatrix} \frac{-1-\sqrt{3}i}{2} \\ \frac{-1+\sqrt{3}i}{2} \\ 1 \end{bmatrix}\right)$.

Then we choose $v_2 = \begin{bmatrix} \frac{-1-\sqrt{3}i}{2\sqrt{3}} \\ \frac{-1+\sqrt{3}i}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ since we need $\|v_3\| = 1$.

Then $\mathcal{B} = \{v_1, v_2, v_3\}$ is the orthonormal basis which make T diagonal:

$$\Big[T\Big]_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{bmatrix} 2 & & \\ & \frac{1+\sqrt{3}i}{2} & \\ & & \frac{1-\sqrt{3}i}{2} \end{bmatrix}.$$