Math 132: Linear algebra II
Spring 2019, Midterm
05/03/2019
Time Limit: 50 Minutes

This exam contains 5 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information in the boxes below. Try your best to write characters as clear as possible. Input exactly one character in each box and don't write anything outside the box.

You are required to show your work on each problem on this exam. The following rules apply:

- You may **NOT** use your books, notes, or any calculator on this exam.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- Please cross (or erase) **everything** you don't want. If both the correct solution and the wrong solution appear simultaneously, we will just grade the wrong one.
- If you need more space, use the last few pages; clearly **indicate** the problem number when you have done this.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- All problems should be answered in **exact values**, not decimal approximations (unless instructed explicitly to do so).
- **DO NOT** write anything outside the box.

1. Please answer the following questions and write examples. Please write down anything you need to make your definitions/examples as clear as possible. You DON'T need to write any proofs.

   (a) (5 points) Please write down the definition of invariant subspaces under a linear operator.

   Solution: Let V be a vector space and $T \in \mathcal{L}(V)$. Let $W \subset V$ be a subspace of V. If for any $w \in W$, $T(w) \in W$, then W is called an invariant subspace of V under T.

   (b) (5 points) Please give ONE example of an invariant subspace W of dimension 2 in a vector space V of dimension 3 under a linear operator $T \in \mathcal{L}(V)$. You should write down explicitly your definition of V, W and T.

   Solution: Let $V = \mathbb{C}^3$ and $\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis. Let $W = \texttt{Span}(e_1, e_2)$. T is defined by

   $$[T]_{\mathcal{S}\leftarrow\mathcal{S}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (c) (5 points) Please write down the definition of eigenvalues and eigenvectors of a linear operator acting on a vector space.

   Solution: Let V be a vector space over $\mathbb{F}$ and $T \in \mathcal{L}(V)$. If there exists $v \in V$ and $\lambda \in \mathbb{F}$ such that $v \neq 0$ and $T(v) = \lambda v$, then λ is called an eigenvalue of T and v is an eigenvector corresponding to the eigenvalue λ.

   (d) (5 points) Please give ONE example of an eigenvalue λ and an eigenvector v of a linear operator T acting on a vector space V of dimension 3. You should write down explicitly your definition of V, T, λ and v.

   Solution: Let $V = \mathbb{C}^3$ and $\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis. Let T be defined by

   $$[T]_{\mathcal{S}\leftarrow\mathcal{S}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Then 1 is an eigenvalue and e_3 is a corresponding eigenvector.

2. (20 points) Suppose V is finite-dimensional, $T, S \in \mathcal{L}(V)$. Let T have $\dim V$ distinct eigenvalues, and S has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

Solution: Since T have $\dim V$ distinct eigenvalues, the eigenvectors of T corresponding to distinct eigenvalues form a basis. Let $v_1, \ldots, v_n$ be these eigenvectors and $\lambda_1, \ldots, \lambda_n$ be corresponding eigenvalues. Then we have $T(v_1) = \lambda_1 v_1, \ldots, T(v_n) = \lambda_n v_n$.

S has the same eigenvectors as T, and we denote the corresponding eigenvalues by $\mu_1, \ldots, \mu_n$. Then we have $S(v_1) = \mu_1 v_1, \ldots, S(v_n) = \mu_n v_n$.

For any $v \in V$, since $\{v_1, \ldots, v_n\}$ is a basis of V, $v = c_1 v_1 + \ldots + c_n v_n$. Then

$$\begin{aligned} TS(v) &= TS(c_1 v_1 + \ldots + c_n v_n) = T(c_1 S(v_1) + \ldots + c_n S(v_n)) = T(c_1 \mu_1 v_1 + \ldots + c_n \mu_n v_n) \\ &= c_1 \mu_1 T(v_1) + \ldots + c_n \mu_n T(v_n) = c_1 \mu_1 \lambda_1 v_1 + \ldots + c_n \mu_n \lambda_n v_n, \end{aligned}$$

and

$$\begin{aligned} ST(v) &= ST(c_1 v_1 + \ldots + c_n v_n) = S(c_1 T(v_1) + \ldots + c_n T(v_n)) = S(c_1 \lambda_1 v_1 + \ldots + c_n \lambda_n v_n) \\ &= c_1 \lambda_1 S(v_1) + \ldots + c_n \lambda_n S(v_n) = c_1 \lambda_1 \mu_1 v_1 + \ldots + c_n \lambda_n \mu_n v_n. \end{aligned}$$

So $TS(v) = ST(v)$. Since this holds for any $v \in V$, $ST = TS$.

3. (20 points) Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\texttt{im}(S)$ is invariant under T.

Solution: For any $v \in \texttt{im}(S)$, there exists $w \in V$ such that $S(w) = v$. Then since $ST = TS$, $T(v) = T(S(w)) = S(T(w))$. So $T(v) \in \texttt{im}(S)$. Then $\texttt{im}(S)$ is invariant under T.

4. (20 points) Suppose $R, T \in \mathcal{L}(\mathbb{C}^3)$. Each have 12, 26, 37 as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbb{C}^3)$ such that $R = S^{-1}TS$.

Solution: Since $R, T \in \mathcal{L}(\mathbb{C}^3)$ each have three distinct eigenvalues, both R and T are diagonalizable. Then there is two bases $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ such that

$$\begin{aligned} R(w_1) = 12 w_1, \quad R(w_2) = 26 w_2, \quad R(w_3) = 37 w_3, \\ T(v_1) = 12 v_1, \quad T(v_2) = 26 v_2, \quad T(v_3) = 37 v_3. \end{aligned}$$

Then let the operator $S \in \mathcal{L}(\mathbb{C}^3)$ be defined by $S(w_i) = v_i$ for $i = 1, 2, 3$. Then for any $a_1, a_2, a_3 \in \mathbb{C}$,

$$S(a_1 w_1 + a_2 w_2 + a_3 w_3) = a_1 S(w_1) + a_2 S(w_2) + a_3 S(w_3) = a_1 v_1 + a_2 v_2 + a_3 v_3.$$

The operator is invertible and the inverse S^{-1} is defined by $S^{-1}(v_i) = w_i$ for $i = 1, 2, 3$. That is, for any $b_1, b_2, b_3 \in \mathbb{C}$,

$$S^{-1}(b_1 v_1 + b_2 v_2 + b_3 v_3) = b_1 S^{-1}(v_1) + b_2 S^{-1}(v_2) + b_3 S^{-1}(v_3) = b_1 w_1 + b_2 w_2 + b_3 w_3.$$

For any $w \in \mathbb{C}^3$, since $\{w_1, w_2, w_3\}$ forms a basis, there exists $a_1, a_2, a_3 \in \mathbb{C}$ such that $w = a_1 w_1 + a_2 w_2 + a_3 w_3$. Then by

$$\begin{aligned} S^{-1}TS(w) &= S^{-1}TS(a_1 w_1 + a_2 w_2 + a_3 w_3) = S^{-1}T(a_1 v_1 + a_2 v_2 + a_3 v_3) \\ &= S^{-1}(a_1 12 v_1 + a_2 26 v_2 + a_3 37 v_3) = a_1 12 w_1 + a_2 26 w_2 + a_3 37 w_3. \end{aligned}$$

and

$$R(w) = R(a_1 w_1 + a_2 w_2 + a_3 w_3) = a_1 R(w_1) + a_2 R(w_2) + a_3 R(w_3) = a_1 12 w_1 + a_2 26 w_2 + a_3 37 w_3,$$

we get $R(w) = S^{-1}TS(w)$. Then $R = S^{-1}TS$.

5. (20 points) Let $V = \mathbb{C}^3$ and $\mathcal{S} = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis. Let $T \in \mathcal{L}(V)$ be defined by the matrix A as $\left[T\right]_{\mathcal{S} \leftarrow \mathcal{S}} = A$:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Please find a basis $\mathcal{B} = \{v_1, v_2, v_3\}$ such that $\left[T\right]_{\mathcal{B} \leftarrow \mathcal{B}}$ is an upper-triangular matrix.

Solution:

1. Find an eigenvector of T on V: $\det(A - \lambda I) = 0$. Then $(2 - \lambda)^3 = 0$. So 2 is the only eigenvalue. Solve the equation $(A - 2I)X = 0$ for $X \in \mathbb{C}^3$.

$$\begin{bmatrix} 2-2 & 1 & 0 \\ 0 & 2-2 & 0 \\ 1 & 0 & 2-2 \end{bmatrix} X = 0.$$

Solutions are $X \in \texttt{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$. We take $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as our first eigenvector.

2. Consider the quotient space $V / \texttt{Span}(v_1)$. Consider the basis $\left\{ [e_1], [e_2] \right\}$ where

$$[e_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \texttt{Span}(v_1), \quad [e_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \texttt{Span}(v_1).$$

$V / \texttt{Span}(v_1) \simeq \mathbb{C}^2$ by $[e_1] \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $[e_2] \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

3. Since

$$\overline{T}([e_1]) = [T(e_1)] = \left[\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right] = 2[e_1],$$

$$\overline{T}([e_2]) = [T(e_2)] = \left[\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right] = [e_1] + 2[e_2].$$

Therefore under the basis $\left\{ [e_1], [e_2] \right\}$, the matrix of $\overline{T}$ is $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

4. Find an eigenvector of $\overline{T}$ on $V / \texttt{Span}(v_1)$: $\det(B - \lambda I) = 0$. Then $(2 - \lambda)^2 = 0$. So 2 is the only eigenvalue. Solve the equation $(B - 2I)X = 0$ for $X \in \mathbb{C}^2 \simeq V / \texttt{Span}(v_1)$.

$$\begin{bmatrix} 2-2 & 1 \\ 0 & 2-2 \end{bmatrix} X = 0.$$

Solutions are $X \in \texttt{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$. We take $\overline{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. By the isomorphism between $\mathbb{C}^2$ and $V / \texttt{Span}(v_1)$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

is corresponding to $\begin{bmatrix}1\\0\\0\end{bmatrix} + \texttt{Span}(v_1)$. We take a representative from the class to be v_2. For example we can pick $v_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

5. Consider the quotient space $V/\,\texttt{Span}(v_1, v_2)$. Consider the basis $\{[e_2]\}$ where

$$[e_2] = \begin{bmatrix}0\\1\\0\end{bmatrix} + \texttt{Span}(v_1, v_2).$$

   $V/\,\texttt{Span}(v_1, v_2) \simeq \mathbb{C}$ by $[e_2] \mapsto 1$.

6. Since

$$\overline{T}([e_2]) = [T(e_2)] = \left[\begin{bmatrix}1\\2\\0\end{bmatrix}\right] = 2[e_2].$$

   Then $[e_2]$ is an eigenvector of $\overline{T}$ in $V/\,\texttt{Span}(v_1, v_2)$. The vector is corresponding to the class $\begin{bmatrix}0\\1\\0\end{bmatrix} + \texttt{Span}(v_1, v_2)$. We take a representative from the class to be v_3. For example we can pick $v_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

7. We now have a basis $\mathcal{B} = \left\{ v_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}, v_3 = \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$. The change-of-basis $P_{\mathcal{S}\leftarrow\mathcal{B}} = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}$. Then the matrix of T is

$$[T]_{\mathcal{B}\leftarrow\mathcal{B}} = P_{\mathcal{S}\leftarrow\mathcal{B}}^{-1}[T]_{\mathcal{S}\leftarrow\mathcal{S}}P_{\mathcal{S}\leftarrow\mathcal{B}} = \begin{bmatrix}2 & 1 & 0\\0 & 2 & 1\\0 & 0 & 2\end{bmatrix}.$$