MATH 138A discussion
University of California, Riverside

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Winter 2019

Final exam solutions

1. Compute the mean, Gaussian, and principal curvatures of the surface $z = x^2 - y^2$ at $(0,0,0)$.

Proof. Consider the parametrized surface

$$\mathbf{x}(u,v) = (u, v, u^2 - v^2).$$

We obtain the first derivatives

$$\begin{aligned}
\mathbf{x}_u(u,v) &= \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial}{\partial u}(u, v, u^2 - v^2) \\
&= \left(\frac{\partial}{\partial u}(u), \frac{\partial}{\partial u}(v), \frac{\partial}{\partial u}(u^2 - v^2)\right) \\
&= (1, 0, 2u)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{x}_v(u,v) &= \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial}{\partial v}(u, v, u^2 - v^2) \\
&= \left(\frac{\partial}{\partial v}(u), \frac{\partial}{\partial v}(v), \frac{\partial}{\partial v}(u^2 - v^2)\right) \\
&= (0, 1, -2v).
\end{aligned}$$

So the coefficients of the first fundamental form are

$$\begin{aligned}
E(u,v) &= \mathbf{x}_u(u,v) \cdot \mathbf{x}_u(u,v) \\
&= (1, 0, 2u) \cdot (1, 0, 2u) \\
&= (1)(1) + (0)(0) + (2u)(2u) \\
&= 1 + 4u^2
\end{aligned}$$

as well as

$$\begin{aligned}
F(u,v) &= \mathbf{x}_u(u,v) \cdot \mathbf{x}_v(u,v) \\
&= (1, 0, 2u) \cdot (0, 1, 2v) \\
&= (1)(0) + (0)(1) + (2u)(-2v) \\
&= -4uv
\end{aligned}$$

and

$$\begin{aligned}
G(u,v) &= \mathbf{x}_v(u,v) \cdot \mathbf{x}_v(u,v) \\
&= (0, 1, 2v) \cdot (0, 1, 2v) \\
&= (0)(0) + (1)(1) + (-2v)(-2v) \\
&= 1 + 4v^2.
\end{aligned}$$

To find the coefficients of the second fundamental form, first we must compute the normal vector. The cross product of $\mathbf{x}_u(u,v)$ and $\mathbf{x}_v(u,v)$ is

$$\begin{aligned}
\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix} \\
&= \begin{vmatrix} 0 & 2u \\ 1 & -2v \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2u \\ 0 & -2v \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\
&= ((0)(-2v) - (1)(2u))\mathbf{i} - ((1)(-2v) - (0)(2u))\mathbf{j} + ((1)(1) - (0)(0))\mathbf{k} \\
&= (-2u)\mathbf{i} + (2v)\mathbf{j} + (1)\mathbf{k} \\
&= (-2u, 2v, 1)
\end{aligned}$$

and its associated magnitude

$$\begin{aligned}
|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)| &= \sqrt{(-2u)^2 + (2v)^2 + (1)^2} \\
&= \sqrt{4u^2 + 4v^2 + 1} \\
&= \sqrt{4(u^2 + v^2) + 1},
\end{aligned}$$

and so the normal vector is

$$\begin{aligned} N(u,v) &= \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)|} \\ &= \frac{(-2u, 2v, 1)}{\sqrt{4(u^2+v^2)+1}}. \end{aligned}$$

Meanwhile, we obtain the second derivatives

$$\begin{aligned} \mathbf{x}_{uu}(u,v) &= \frac{\partial \mathbf{x}_u}{\partial u} = \frac{\partial}{\partial u}(1, 0, 2u) \\ &= \left(\frac{\partial}{\partial u}(1), \frac{\partial}{\partial u}(0), \frac{\partial}{\partial u}(2u)\right) \\ &= (0, 0, 2), \end{aligned}$$

as well as

$$\begin{aligned} \mathbf{x}_{uv}(u,v) &= \frac{\partial \mathbf{x}_u}{\partial v} = \frac{\partial}{\partial v}(1, 0, 2u) \\ &= \left(\frac{\partial}{\partial v}(1), \frac{\partial}{\partial v}(0), \frac{\partial}{\partial v}(2u)\right) \\ &= (0, 0, 0) \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_{vv}(u,v) &= \frac{\partial \mathbf{x}_v}{\partial v} = \frac{\partial}{\partial v}(0, 1, -2v) \\ &= \left(\frac{\partial}{\partial v}(0), \frac{\partial}{\partial v}(1), \frac{\partial}{\partial v}(-2v)\right) \\ &= (0, 0, -2). \end{aligned}$$

So the coefficients of the second fundamental form are

$$\begin{aligned} e(u,v) &= N(u,v) \cdot \mathbf{x}_{uu}(u,v) \\ &= \frac{(-2u, 2v, 1)}{\sqrt{4(u^2+v^2)+1}} \cdot (0, 0, 2) \\ &= \frac{(-2u)(0) + (2v)(0) + (1)(2)}{\sqrt{4(u^2+v^2)+1}} \\ &= \frac{2}{\sqrt{4(u^2+v^2)+1}}, \end{aligned}$$

as well as

$$\begin{aligned} f(u,v) &= N(u,v) \cdot \mathbf{x}_{uv}(u,v) \\ &= \frac{(-2u, 2v, 1)}{\sqrt{4(u^2+v^2)+1}} \cdot (0, 0, 0) \\ &= \frac{(-2u)(0) + (-2v)(0) + (1)(0)}{\sqrt{4(u^2+v^2)+1}} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} g(u,v) &= N(u,v) \cdot \mathbf{x}_{uu}(u,v) \\ &= \frac{(-2u, 2v, 1)}{\sqrt{4(u^2+v^2)+1}} \cdot (0, 0, -2) \\ &= \frac{(-2u)(0) + (2v)(0) + (1)(-2)}{\sqrt{4(u^2+v^2)+1}} \\ &= -\frac{2}{\sqrt{4(u^2+v^2)+1}}. \end{aligned}$$

Using the formula on page 155 of do Carmo, the Gaussian curvature is

$$\begin{aligned}
K(u,v) &= \frac{eg-f^2}{EG-F^2} \\
&= \frac{(\frac{2}{\sqrt{4(u^2+v^2)+1}})(-\frac{2}{\sqrt{4(u^2+v^2)+1}}) - (0)^2}{(1+4u^2)(1+4v^2)-(4uv)^2} \\
&= -\frac{\frac{4}{4(u^2+v^2)+1}}{(1+4(u^2+v^2)+16u^2v^2)-16u^2v^2} \\
&= -\frac{4}{(1+4(u^2+v^2))^2}.
\end{aligned}$$

Using the formula on page 156 of do Carmo, the mean curvature is

$$\begin{aligned}
H(u,v) &= \frac{1}{2}\frac{eG-2fF+gE}{EG-F^2} \\
&= \frac{1}{2}\frac{(\frac{2}{\sqrt{4(u^2+v^2)+1}})(1+4v^2) - 2(0)(4uv) + (-\frac{2}{\sqrt{4(u^2+v^2)+1}})(1+4u^2)}{(1+4u^2)(1+4v^2)-(4uv)^2} \\
&= \frac{1}{2}\frac{\frac{2}{\sqrt{4(u^2+v^2)+1}}(4(v^2-u^2))}{(1+4(u^2+v^2)+16u^2v^2)-16u^2v^2} \\
&= \frac{4(v^2-u^2)}{(1+4(u^2+v^2))^{\frac{3}{2}}}.
\end{aligned}$$

We will now compute the principal curvatures k_1, k_2. To do this, we will solve for k_1, k_2 from the formulas $H = \frac{k_1+k_2}{2}$ and $K = k_1k_2$. From $H = \frac{k_1+k_2}{2}$, we get $k_1 = 2H - k_2$, and so we get

$$\begin{aligned}
K &= k_1k_2 \\
&= (2H-k_2)k_2 \\
&= 2Hk_2 - k_2^2,
\end{aligned}$$

which is algebraically equivalent to the quadratic equation

$$k_2^2 - 2Hk_2 + K = 0.$$

Employing the quadratic formula, we get

$$\begin{aligned}
k_2 &= \frac{-(-2H) \pm \sqrt{(-2H)^2 - 4(1)(K)}}{2(1)} \\
&= \frac{2H \pm \sqrt{4(H^2-K)}}{2} \\
&= H \pm \sqrt{H^2-K}.
\end{aligned}$$

This also means

$$\begin{aligned}
k_1 &= 2H - k_2 \\
&= 2H - (H \pm \sqrt{H^2-K}) \\
&= H \mp \sqrt{H^2-K}.
\end{aligned}$$

As we conventionally require $k_1 > k_2$, we will choose $k_1 = H + \sqrt{H^2-K}$ and $k_2 = H - \sqrt{H^2-K}$. Substituting our expressions for H, K, our principal curvatures are

$$\begin{aligned}
k_1(u,v) &= H(u,v) + \sqrt{H(u,v)^2 - K(u,v)} \\
&= \left(\frac{4(v^2-u^2)}{(1+4(u^2+v^2))^{\frac{3}{2}}}\right) + \sqrt{\left(\frac{4(v^2-u^2)}{(1+4(u^2+v^2))^{\frac{3}{2}}}\right)^2 - \left(-\frac{4}{(1+4(u^2+v^2))^2}\right)^2} \\
&= \frac{4(v^2-u^2)}{(1+4(u^2+v^2))^{\frac{3}{2}}} + \sqrt{\frac{16(v^2-u^2)^2}{(1+4(u^2+v^2))^3} + \frac{16}{(1+4(u^2+v^2))^4}}
\end{aligned}$$

and

$$\begin{aligned}
k_2(u,v) &= H(u,v) - \sqrt{H(u,v)^2 - K(u,v)} \\
&= \left(\frac{4(v^2-u^2)}{(1+4(u^2+v^2))^{\frac{3}{2}}}\right) - \sqrt{\left(\frac{4(v^2-u^2)}{(1+4(u^2+v^2))^{\frac{3}{2}}}\right)^2 - \left(-\frac{4}{(1+4(u^2+v^2))^2}\right)^2} \\
&= \frac{4(v^2-u^2)}{(1+4(u^2+v^2))^{\frac{3}{2}}} - \sqrt{\frac{16(v^2-u^2)^2}{(1+4(u^2+v^2))^3} + \frac{16}{(1+4(u^2+v^2))^4}}.
\end{aligned}$$

(Note: The general formulas for principal curvatures $k_1(u,v), k_2(u,v)$ we just derived here are *completely optional*; they are here but you do not have to find the general formulas in the first place in order to find $k_1(0,0), k_2(0,0)$.) At the origin $(0,0,0)$, our original parametrization $\mathbf{x}(u,v) = (u,v,u^2-v^2)$ implies $u=0$ and $v=0$. So, at the origin, the Gaussian curvature is

$$\begin{aligned}
K(0,0) &= -\frac{4}{(1+4(0^2+0^2))^2} \\
&= -4
\end{aligned}$$

and the mean curvature is

$$\begin{aligned}
H(0,0) &= \frac{4(0^2-0^2)}{(1+4(0^2+0^2))^{\frac{3}{2}}} \\
&= 0,
\end{aligned}$$

and so

$$\begin{aligned}
k_1(0,0) &= H(0,0) + \sqrt{H(0,0)^2 - K(0,0)} \\
&= 0 + \sqrt{0^2-(-4)} \\
&= 0+2 \\
&= 2
\end{aligned}$$

and

$$\begin{aligned}
k_2(0,0) &= H(0,0) - \sqrt{H(0,0)^2 - K(0,0)} \\
&= 0 - \sqrt{0^2-(-4)} \\
&= 0-2 \\
&= -2,
\end{aligned}$$

are our principal curvatures at the origin. □

2. State and prove the Meusnier Theorem.

Proof. Statement of Meusnier Theorem: All curves lying on a surface S and having at a given point $p \in S$ the same tangent line have at this point the same normal curvatures (c.f. Proposition 2 of Section 3.2; c.f. do Carmo, page 142).

Proof of Meusnier Theorem: Following page 142 of do Carmo, we will instead prove the more general claim: The value of the second fundamenatal form II_p for a unit vector $v \in T_p(S)$ is equal to the normal curvature of a regular curve passing through p and tangent to v, i.e. $\mathrm{II}_p(\alpha(0)) = k_n(p)$. Once we do this, Meusnier's Theorem will follow. To prove our claim, let C be a regular curve in the surface S parametrized by $\alpha(s)$, which satisfies $\alpha(0) = p$, where s is the arc length of C. Let $N(s)$ be the restriction of the normal vector N defined on S to the curve $\alpha(s)$. Then we have $N(s)\cdot\alpha'(s) = 0$, from which we can take the derivatives in s of both sides to obtain

$$N'(s)\cdot\alpha'(s) + N(s)\cdot\alpha''(s) = 0,$$

or

$$N(s)\cdot\alpha''(s) = -N'(s)\cdot\alpha'(s).$$

Therefore, using this and a Frenet formula, we can conclude

$$\begin{aligned}
\mathrm{II}_p(\alpha'(0)) &= -dN_p(\alpha'(0))\cdot\alpha'(0) \\
&= -N'(0)\cdot\alpha'(0) \\
&= N(0)\cdot\alpha''(0) \\
&= N(0)\cdot t'(0) \\
&= N(0)\cdot kn(0) \\
&= (N\cdot kn)(p) \\
&= k_n(p),
\end{aligned}$$

which proves our claim. □

3. Let S be a connected regular surface in $\mathbb{R}^3$ that is umbilical at every point. Prove that the Gaussian curvature of S is constant.

Proof. We recall that a point $p \in S$ is said to be umbilical if the principal curvatures k_1, k_2 of S satisfy $k_1 = k_2$ at p. To prove that the Gaussian curvature of S is constant, it suffices to prove that S is contained in the sphere S^2 of radius $r > 0$ or in a plane such as $\mathbb{R}^2$, for they have constant Gaussian curvatures $K_{\mathbb{R}^2} = 0$ and $K_{S^2} = \frac{1}{r^2}$, respectively. This reduces our goal to proving Proposition 4 of Section 3-2 in do Carmo (c.f. pages 147-148); we will now follow the proof. Let $\mathbf{x}(u, v)$ be a parametrization such that the coordinate neighborhood $V \subset S$ containing p is connected. Since each $q \in V$ is an umbilical point, for any vector $w \in T_q(S)$, which we can write

$$w(u, v) = a_1(u, v)\mathbf{x}_u(u, v) + a_2\mathbf{x}_v(u, v)$$

as a local coordinate expression in V, we have

$$dN(w) = \lambda(q)w,$$

where $\lambda = \lambda(q)$ is a real differentiable function in V. We first show that $\lambda(q)$ is constant in V. Using local coordinates in V, our above equation gives us

$$\begin{aligned} N_u(u, v)a_1 + N_v(u, v)a_2 &= dN(w) \\ &= \lambda(q)w \\ &= \lambda(q)(\mathbf{x}_u(u, v)a_1 + \mathbf{x}_u(u, v)a_2) \\ &= \lambda(q)\mathbf{x}_u(u, v)a_1 + \lambda(q)\mathbf{x}_u(u, v)a_2. \end{aligned}$$

Hence, since w is arbitrary, we can equate the terms (or, rather, equate the coefficients) to conclude that our partial derivatives of N are

$$\begin{aligned} N_u(u, v) &= \lambda(q)\mathbf{x}_u(u, v), \\ N_v(u, v) &= \lambda(q)\mathbf{x}_v(u, v). \end{aligned}$$

If we perform partial differentiations on both sides of the first equation $N_u(u, v) = \lambda(q)\mathbf{x}_u(u, v)$ with respect to v and both sides of the second equation $N_v(u, v) = \lambda(q)\mathbf{x}_v(u, v)$ with respect to u, then we get

$$\begin{aligned} N_{uv}(u, v) &= \lambda_v(q)\mathbf{x}_u(u, v) + \lambda(q)\mathbf{x}_{uv}(u, v), \\ N_{vu}(u, v) &= \lambda_u(q)\mathbf{x}_v(u, v) + \lambda(q)\mathbf{x}_{vu}(u, v), \end{aligned}$$

from which we conclude

$$\begin{aligned} \lambda_u(q)\mathbf{x}_v(u, v) - \lambda_v(q)\mathbf{x}_u(u, v) &= (N_{uv}(u, v) - \lambda(q)\mathbf{x}_{uv}(u, v)) - (N_{vu}(u, v) - \lambda(q)\mathbf{x}_{vu}(u, v)) \\ &= N_{uv}(u, v) - N_{vu}(u, v) + \lambda(q)(\mathbf{x}_{vu}(u, v) - \mathbf{x}_{uv}(u, v)) \\ &= 0. \end{aligned}$$

In fact, since $\mathbf{x}_u, \mathbf{x}_v$ are linearly independent vectors, we conclude that the coefficients λ_u, λ_v must be zero, i.e. $\lambda_u = 0$ and $\lambda_v = 0$, for all $q \in V$. The zero partial derivatives of λ therefore suggest that λ is constant on V, since V is connected. Now, we must deal with two cases of our constant λ separately. For the first case, if $\lambda \equiv 0$, then $N_u = \lambda\mathbf{x}_u \equiv 0\mathbf{x}_u = 0$ and $N_v = \lambda\mathbf{x}_v \equiv 0\mathbf{x}_v = 0$, and so N must be constant on V, say $N = N_0$ for some constant vector on V. This means

$$\begin{aligned} \frac{\partial}{\partial u}(\mathbf{x}(u, v) \cdot N_0) &= \mathbf{x}_u(u, v) \cdot N_0 \\ &= 0 \cdot N_0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial v}(\mathbf{x}(u, v) \cdot N_0) &= \mathbf{x}_v(u, v) \cdot N_0 \\ &= 0 \cdot N_0 \\ &= 0. \end{aligned}$$

Hence, $\mathbf{x}(u, v) \cdot N_0$ is constant, and so all the points $\mathbf{x}(u, v)$ on V are contained in a plane. For the second case, if $\lambda \neq 0$, then $\frac{1}{\lambda}$ is well-defined, which means we can have

$$\begin{aligned} \frac{\partial}{\partial u}\left(\mathbf{x}(u, v) - \frac{1}{\lambda}N(u, v)\right) &= \mathbf{x}_u(u, v) - \frac{1}{\lambda}N_u(u, v) \\ &= 0 - \frac{1}{\lambda}0 \\ &= 0 \end{aligned}$$

and

$$\frac{\partial}{\partial v}\left(\mathbf{x}(u,v) - \frac{1}{\lambda}N(u,v)\right) = \mathbf{x}_v(u,v) - \frac{1}{\lambda}N_v(u,v)$$
$$= 0 - \frac{1}{\lambda}0$$
$$= 0.$$

These two statements imply that the point

$$\mathbf{y}(u,v) := \mathbf{x}(u,v) - \frac{1}{\lambda}N(u,v)$$

is constant in $(u,v) \in V$, i.e. fixed on V. Hence,

$$|\mathbf{x}(u,v) - \mathbf{y}(u,v)| = \left|\frac{1}{\lambda}N(u,v)\right|$$
$$= \frac{1}{|\lambda|}|N(u,v)|$$
$$= \frac{1}{|\lambda|}(1)$$
$$= \frac{1}{|\lambda|},$$

and so all points of V are contained in a sphere of center $\mathbf{y}(u,v)$ and radius $\frac{1}{|\lambda|}$. □

4. Let $\alpha(s)$ be a unit speed curve in $\mathbb{R}^2$. Prove that the torsion $\tau(s)$ of the curve is zero if and only if the curve is planar.

Proof. We will prove the forward direction: If the torsion $\tau(s)$ of the curve $\alpha(s)$ is zero (i.e. $\tau \equiv 0$), then the curve is planar. Since we assumed $\tau \equiv 0$, we must have

$$b'(s) = \tau(s)n(s)$$
$$= 0n(s)$$
$$= 0,$$

and so $b(s)$ is constant, i.e. $b(s) = b_0$ for some fixed vector b_0. Therefore,

$$\frac{d}{ds}(\alpha(s)\cdot b_0) = \alpha'(s)\cdot b_0$$
$$= 0\cdot b_0$$
$$= 0,$$

from which we conclude that $\alpha(s)\cdot b_0$ is constant, and so $\alpha(s)$ is contained in a plane normal to b_0.

Now, we will prove the backward direction: If our curve $\alpha(s)$ is planar, then $\tau \equiv 0$. Since $\alpha(s)$ is planar for all $s \in I$ (i.e. $\alpha(I)$ is contained in a plane), it follows that the plane containing our curve agrees with the osculating plane. And any curve in an osculating plane must have zero torsion; in particular, our curve $\alpha(s)$ satisfies $\tau \equiv 0$.

(This proof is taken from Section 1-5 of do Carmo; c.f. page 18.) □

5. a. State the Isoperimetric Inequality.

   Proof. Let C be a simple closed plane curve with length l, and let A be the area of the region bounded by C. Then

   $$l^2 - 4\pi A \geq 0,$$

   and equality holds if and only if C is a circle. (This is Theorem 1 of Section 1-7 of do Carmo; c.f. page 33). □

   b. Is there a simple closed curve C in the plane with length equal to 5 feet bounding an area of 2 square feet?

   Proof. Since C is a simply closed curve of some length l and bounding some area A, we must have the Isoperimetric Inequality $l^2 - 4\pi A \geq 0$ (c.f. do Carmo, page 33). However, if $l = 5$ and $A = 2$, then

   $$l^2 - 4\pi A = (5)^2 - 4\pi(2)$$
   $$= 25 - 8\pi$$
   $$< 0.$$

   So the Isoperimetric Inequality is not satisfied, which means there does not exist such a simple closed curve with $l = 5$, $A = 2$. □

c. In part b, what is the maximum area that C can bound? What is this curve?

Proof. We can algebraically rearrange the inequality $l^2 - 4\pi A \geq 0$ to find an upper bound of the area:

$$A \leq \frac{l^2}{4\pi},$$

from which it is easier to see that the maximum area is

$$A_{\max} = \frac{l^2}{4\pi}.$$

Since we were given that our curve C has length $l = 5$, the maximum area A that C can bound is

$$\begin{aligned} A_{\max} &= \frac{l^2}{4\pi} \\ &= \frac{(5)^2}{4\pi} \\ &= \frac{25}{4\pi}. \end{aligned}$$

Furthermore, the statement of the Isoperimetric Inequality from part a asserts that C must be a circle. □

6. Let S be a compact regular surface in $\mathbb{R}^3$. Prove that its mean curvature cannot vanish everywhere. (Since S with vanishing mean curvature is really another way of saying that S is minimal, this question is the same as Exercise 3-5.12 of do Carmo, and the solution to that exercise is in Homework 7.)

Proof. Suppose to the contrary that there exists some surface $S \subset \mathbb{R}^3$ whose curvature vanishes everywhere, i.e. $H \equiv 0$. So we have

$$\begin{aligned} 0 &= H \\ &= \frac{k_1 + k_2}{2} \end{aligned}$$

which implies that k_1, k_2 have opposite signs. Consequently, we have

$$\begin{aligned} \det(dN_p) &= K \\ &= k_1 k_2 \\ &< 0 \end{aligned}$$

for any arbitrary point $p \in S$, which implies that S does not have any elliptic points. But this contradicts Exercise 3-3.16, which asserts that S has an elliptic point since we also assumed that S is compact. Therefore, no compact minimal surfaces exist in $\mathbb{R}^3$.

It remains to prove Exercise 3-3.16 (whose solution is also found in Homework 7). Let $p \in \mathbb{R}^3$ be an elliptic point, which means $\det(dN_p) > 0$ (c.f. do Carmo, page 146), where we recall that dN_p is the differential of the Gauss map N_p. Now, let S be a compact surface. Then there exists a sphere of a sufficiently large radius $R > 0$ such that S lies inside of the sphere, except at only one point—call it p—that touches the sphere. (Note: it would be helpful to draw a picture of this.) Let K_S and K_{S^2} denote respectively the Gaussian curvatures of the surface $S \subset \mathbb{R}^3$ and of the sphere $S^2 \subset \mathbb{R}^3$ at the point p. Then $K_{S^2} = \frac{1}{R^2} > 0$ for some large enough $R > 0$, where R is the radius of the sphere S^2. Also, $K_S \geq K_{S^2}$ at the point p, since S is contained inside S^2. Therefore,

$$\begin{aligned} \det(dN_p) &= K_S \\ &\geq K_{S^2} \\ &= \frac{1}{R^2} \\ &> 0, \end{aligned}$$

which means p is an elliptic point. □

7. a. State the fundamental theorem for the local theory of curves.

Proof. Given differentiable functions $k(s) > 0$ and $\tau(s)$ for all $s \in I$, where I is an interval in $\mathbb{R}$, there exists a regular parametrized curve $\alpha : I \longrightarrow \mathbb{R}^3$ such that s is the arc length, $k(s)$ is the curvature, and $\tau(s)$ is the torsion of α. Moreover, any other curve $\bar{\alpha}$ satisfying the same conditions differs from α by a rigid motion; that is, there exists an orthogonal linear map ρ of $\mathbb{R}^3$, with positive determinant, and a vector c such that $\bar{\alpha} = \rho \circ \alpha + c$. (This is in Section 1-5 of do Carmo; c.f. page 19.) □

b. Prove the uniqueness part (i.e. rigidity theorem) of part a.

Proof. (The following proof is taken from Section 1-5 of do Carmo, c.f. pages 20-21, although for clarity I added a couple additional steps to the calculations.) Assume that two curves $\alpha = \alpha(s)$ and $\bar{\alpha} = \bar{\alpha}(s)$ satisfy the conditions $k(s) = \bar{k}(s)$ and $\tau(s) = \bar{\tau}(s)$ for all $s \in I$. Let t_0, n_0, b_0 and $\bar{t}_0, \bar{n}_0, \bar{b}_0$ be the Frenet trihedrons at $s_0 \in I$ of α and $\bar{\alpha}$, respectively. Then there is a rigid motion which sends $\bar{\alpha}(s_0)$ into $\alpha(s_0)$ and $\bar{t}_0, \bar{n}_0, \bar{b}_0$ into t_0, n_0, b_0, respectively. Thus, after performing this rigid motion on $\bar{\alpha}$, we have that $\alpha(s_0) = \alpha(s_0)$ and that the Frenet trihedrons $t(s), n(s), b(s)$ and $\bar{t}(s), \bar{n}(s), \bar{b}(s)$ of α and $\bar{\alpha}$, respectively, satisfy the Frenet equations

$$\begin{aligned} \frac{dt}{ds} &= kn & \frac{d\bar{t}}{ds} &= k\bar{n} \\ \frac{dn}{ds} &= -kt - \tau b & \frac{d\bar{n}}{ds} &= -k\bar{t} - \tau\bar{n} \\ \frac{db}{ds} &= \tau n & \frac{d\bar{b}}{ds} &= \tau\bar{n}, \end{aligned}$$

with $t(s_0) = \bar{t}(s_0)$, $n(s_0) = \bar{n}(s_0)$, $b(s_0) = \bar{b}(s_0)$. We now observe, by using the Frenet equations, that

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}(|t-\bar{t}|^2 + |n-\bar{n}|^2 + |b-\bar{b}|^2) &= (t-\bar{t})\cdot(t'-\bar{t}') + (b-\bar{b})\cdot(b'-\bar{b}') + (n-\bar{n})\cdot(n'-\bar{n}') \\ &= (t-\bar{t})\cdot(kn-k\bar{n}) + (b-\bar{b})\cdot(\tau n - \tau n') + (n-\bar{n})\cdot((-kt-\tau b)-(-k\bar{t}-\tau\bar{b})) \\ &= k(t-\bar{t})\cdot(n-\bar{n}) + \tau(b-\bar{b})\cdot(n-\bar{n}) - k(n-\bar{n})\cdot(t-\bar{t}) - \tau(n-\bar{n})\cdot(b-\bar{b}) \\ &= k(t-\bar{t})\cdot(n-\bar{n}) - k(n-\bar{n})\cdot(t-\bar{t}) + \tau(b-\bar{b})\cdot(n-\bar{n}) - \tau(n-\bar{n})\cdot(b-\bar{b}) \\ &= 0 \end{aligned}$$

for all $s \in I$. Thus, the expression $|t-\bar{t}|^2 + |n-\bar{n}|^2 + |b-\bar{b}|^2$ is constant in s, and, since it is zero at $s = s_0$, we conclude that the expression must be identically zero, i.e.

$$|t-\bar{t}|^2 + |n-\bar{n}|^2 + |b-\bar{b}|^2 \equiv 0,$$

from which we get $|t-\bar{t}| = |n-\bar{n}| = |b-\bar{b}| = 0$, and so it follows that $t(s) = \bar{t}(s)$, $n(s) = \bar{n}(s)$, $b(s) = \bar{b}(s)$ for all $s \in I$. Since we also have

$$\frac{d\alpha}{ds} = t = \bar{t} = \frac{d\bar{\alpha}}{ds},$$

we obtain

$$\begin{aligned} \frac{d}{ds}(\alpha(s) - \bar{\alpha}(s)) &= \bar{\alpha}'(s) - \alpha'(s) \\ &= \bar{t}(s) - t(s) \\ &= 0. \end{aligned}$$

Thus, $\bar{\alpha}(s) - \alpha(s)$ is constant, i.e. $\bar{\alpha}(s) - \alpha(s) = a$, or

$$\alpha(s) = \bar{\alpha}(s) + a,$$

where a is a constant vector in $\mathbb{R}^3$. Since $\alpha(s_0) = \bar{\alpha}(s_0)$, we must have $a = 0$. Hence,

$$\alpha(s) = \bar{\alpha}(s)$$

for all $s \in I$. □

8. Show that the mean curvature H at $p \in S$ of a regular surface in $\mathbb{R}^3$ is given by

$$H = \frac{1}{\pi}\int_0^\pi k_n(\theta)\,d\theta,$$

where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction. (This is Exercise 3-2.5 of do Carmo, and the solution to that exercise is in Homework 6.)

Proof. The normal curvature k_n is given by Euler's formula (c.f. do Carmo, page 145)

$$k_n(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

We also recall that the mean curvature is given by (c.f. do Carmo, page 146)

$$H = \frac{k_1 + k_2}{2}.$$

So we have

$$\begin{aligned}
\int_0^\pi k_n(\theta)\,d\theta &= \int_0^\pi k_1\cos^2\theta + k_2\sin^2\theta\,d\theta \\
&= \int_0^\pi k_1\frac{1+\cos(2\theta)}{2} + k_2\frac{1-\cos(2\theta)}{2}\,d\theta \\
&= \frac{k_1}{2}\int_0^\pi 1+\cos(2\theta)\,d\theta + \frac{k_2}{2}\int_0^\pi 1-\cos(2\theta)\,d\theta \\
&= \frac{k_1}{2}\left(\theta+\frac{1}{2}\sin(2\theta)\right)\Bigg|_0^\pi + \frac{k_2}{2}\left(\theta-\frac{1}{2}\sin(2\theta)\right)\Bigg|_0^\pi \\
&= \frac{k_1}{2}\left(\left(\pi+\frac{1}{2}\sin(2\pi)\right)-\left(0+\frac{1}{2}\sin(2(0))\right)\right) + \frac{k_2}{2}\left(\left(\pi-\frac{1}{2}\sin(2\pi)\right)-\left(0-\frac{1}{2}\sin(2(0))\right)\right) \\
&= \frac{k_1}{2}\pi + \frac{k_2}{2}\pi \\
&= \pi\frac{k_1+k_2}{2} \\
&= \pi H,
\end{aligned}$$

which implies algebraically

$$H = \frac{1}{\pi}\int_0^\pi k_n(\theta)\,d\theta$$

as desired. □