MATH 138A discussion
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Solutions to assigned homework problems from *Differential Geometry of Curves and Surfaces* by Manfredo Perdigão do Carmo

Assignment 1 – page 5: 1,2,4,5 and page 7: 1,2

1-2.1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

Proof. An example of such a curve is

$$\alpha(t) = \left(\cos\left(\frac{\pi}{2} - t\right), \sin\left(\frac{\pi}{2} - t\right)\right).$$

Indeed, $\alpha(0) = (\cos\frac{\pi}{2}, \sin\frac{\pi}{2}) = (0, 1)$ and $\alpha(t)$ travels clockwise in the unit circle whenever t is increasing. □

1-2.2. Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Consider the function $f(t) = |\alpha(t)|^2$, which measures the distance between the origin and $\alpha(t)$ for all $t \in I$. If $t_0 \in I$ is a value at which $\alpha(t_0)$ is the point of the trace of $\alpha(t)$ closest to the origin, then the distance is minimzed, which means $f(t)$ is at its minimum at t_0, and so $f'(t) = 0$. But we also have

$$\begin{aligned} f'(t) = \frac{df}{dt} &= \frac{d}{dt}(|\alpha(t)|^2) \\ &= \frac{d}{dt}(\alpha(t) \cdot \alpha(t)) \\ &= \alpha'(t) \cdot \alpha(t) + \alpha(t) \cdot \alpha'(t) \\ &= \alpha'(t) \cdot \alpha(t) + \alpha'(t) \cdot \alpha(t) \\ &= 2\alpha'(t) \cdot \alpha(t). \end{aligned}$$

In particular, we have $f'(t_0) = 2\alpha'(t_0) \cdot \alpha(t_0)$. Equating both expression of $f'(t_0)$, we conclude that $\alpha(t_0) \cdot \alpha(t_0) = 0$, which means $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$. □

1-2.4. Let $\alpha : I \to \mathbb{R}^3$ be a parametrized cure and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Proof. Since $\alpha'(t)$ is orthogonal to v, we have $\alpha'(t) \cdot v = 0$. Since v is a fixed vector, it does not depend on t, which means $\frac{dv}{dt} = 0$. So

$$\begin{aligned} \frac{d}{dt}(\alpha(t) \cdot v) &= \frac{d\alpha}{dt} \cdot v + \alpha(t) \cdot \frac{dv}{dt} \\ &= \alpha'(t) \cdot v + \alpha(t) \cdot 0 \\ &= \alpha'(t) \cdot v \\ &= 0. \end{aligned}$$

Hence, $\alpha(t) \cdot v$ is constant in t. In particular, $\alpha(t) \cdot v = \alpha(0) \cdot v = 0$, which means $\alpha(t)$ is perpendicular to v, for all $t \in I$. □

1-2.5. Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Proof. Suppose $|\alpha(t)| = \text{const.}$. Then

$$\begin{aligned} \frac{d}{dt}(\alpha(t) \cdot \alpha(t)) &= \frac{d}{dt}(|\alpha(t)|^2) \\ &= \frac{d}{dt}(\text{const.})^2 \\ &= 0. \end{aligned}$$

At the same time, we have

$$\begin{aligned} \frac{d}{dt}(\alpha(t) \cdot \alpha(t)) &= \alpha'(t) \cdot \alpha(t) + \alpha(t) \cdot \alpha'(t) \\ &= \alpha(t) \cdot \alpha'(t) + \alpha(t) \cdot \alpha'(t) \\ &= 2\alpha(t) \cdot \alpha'(t). \end{aligned}$$

Equating the two expressions, we conclude that $\alpha(t) \cdot \alpha'(t) = 0$; that is, $\alpha(t)$ is orthogonal to $\alpha(t)$.

Conversely, suppose $\alpha(t)$ is orthogonal to $\alpha'(t)$; that is, $\alpha(t) \cdot \alpha'(t) = 0$. Then by the calculations of the previous paragraph, we have $|\alpha(t)| = \text{const.}$. Now we need to show that the constant is nonzero. Instead, suppose to the contrary that the constant is zero. Then $|\alpha(t)| = 0$, which means $\alpha(t) = 0$ for all $t \in I$. This implies that $\alpha'(t) = 0$ for all $t \in I$, which contradicts one of the hypotheses which states $\alpha'(t) \neq 0$ for all $t \in I$. So we must have that the constant be nonzero; that is, $|\alpha(t)|$ is a nonzero constant. □

1-3.1. Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0$, $z = x$.

Proof. Let θ be the angle between $\alpha'(t)$ and the line described by $y = 0$ and $z = x$. Also consider the vectors $(1, 0, 1)$ and $(-1, 0, -1)$, both of which lie in the line $y = 0$, $z = x$. For convenience in our calculations, we will describe $(1, 0, 1)$ and $(-1, 0, -1)$ simultaneously as $(\pm 1, 0, \pm 1)$. These two vectors have the same magnitude $|(\pm 1, 0, \pm 1)| = \sqrt{(\pm 1)^2 + (0)^2 + (\pm 1)^2} = \sqrt{2}$. Now considering the parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$, We obtain the derivative

$$\begin{aligned}
\alpha'(t) &= \frac{d\alpha}{dt} = \frac{d}{dt}(3t, 3t^2, 2t^3) \\
&= \left(\frac{d}{dt}(3t), \frac{d}{dt}(3t^2), \frac{d}{dt}(2t^3)\right) \\
&= (3, 6t, 6t^2)
\end{aligned}$$

and its associated magnitude

$$\begin{aligned}
|\alpha'(t)| &= \sqrt{(3)^2 + (6t)^2 + (6t^2)^2} \\
&= \sqrt{9 + 36t^2 + 36t^4} \\
&= \sqrt{9(1 + 4t^2 + 4t^4)} \\
&= \sqrt{9(1 + 2t^2)^2} \\
&= \sqrt{9}\sqrt{(1 + 2t^2)^2} \\
&= 3(1 + 2t^2).
\end{aligned}$$

So we have

$$\begin{aligned}
\cos\theta &= \frac{\alpha'(t) \cdot (\pm 1, 0, \pm 1)}{|\alpha'(t)||(\pm 1, 0, \pm 1)|} \\
&= \frac{(3, 6t, 6t^2) \cdot (\pm 1, 0, \pm 1)}{(3(1 + 2t^2))(\sqrt{2})} \\
&= \frac{(3)(\pm 1) + (6t)(0) + (6t^2)(\pm 1)}{3\sqrt{2}(1 + 2t^2)} \\
&= \frac{\pm 3 \pm 6t^2}{3\sqrt{2}(1 + 2t^2)} \\
&= \frac{\pm 3(1 + 2t^2)}{3\sqrt{2}(1 + 2t^2)} \\
&= \pm\frac{1}{\sqrt{2}},
\end{aligned}$$

and so $\theta = \cos(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$ or $\theta = \cos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$. Since neither of these two values depends on s, we conclude that θ is a constant angle (that only depends on the choice of one of the two vectors $(1, 0, 1)$ or $(-1, 0, -1)$ in the line $y = 0$, $z = x$). □

1-3.2. A circular disk of radius 1 in the plane xy rolls without slipping along the x-axis. The figure described by a point of the circumference of the disk is called a cycloid.

a. Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$, the trace of which is the cycloid, and determine its singular (critical) points.

Proof. Since the circular disk rolls to the right without slipping, the horizontal distance traveled by the wheel is equal to the arc length of the circle. In other words, since the given angle at time t is t radians, it follows that the arc length is $r\theta = (1)(t) = t$, which means the horizontal distance from the origin at time t is t as well. Moreover, since the radius of the circular disk is 1, the height of the center from the x-axis is 1. Therefore, the (x, y)-coordinate of the center of the circular disk is $(t, 1)$. Now, the point on the edge of the circular disk is located initially at $-\frac{\pi}{2}$ radians, and the point rotates clockwise by t radians at time t. So the clockwise motion of the point on the edge of the circular disk is described by $(\cos(-\frac{\pi}{2}-t), \sin(-\frac{\pi}{2}-t))$. Therefore, the overall motion of the point on the bottom at the circular disk initially on the origin is parametrized by

$$\alpha(t) := (t, 1) + \left(\cos\left(-\frac{\pi}{2}-t\right), \sin\left(-\frac{\pi}{2}-t\right)\right).$$

Furthermore, since sine is an odd function and cosine is an even function (i.e. $\sin(-t) = -\sin t$ and $\cos(-t) = \cos t$, and using the trigonometric identities $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and $\cos(a+b) = \cos a \cos b - \sin a \sin b$, we have

$$\begin{aligned}\left(\cos\left(-\frac{\pi}{2}-t\right), \sin\left(-\frac{\pi}{2}-t\right)\right) &= \left(\cos\left(-\frac{\pi}{2}\right)\cos(-t) - \sin\left(-\frac{\pi}{2}\right)\sin(-t), \sin\left(-\frac{\pi}{2}\right)\cos(-t) + \cos\left(-\frac{\pi}{2}\right)\sin(-t)\right) \\ &= (0\cos(-t) - (-1)\sin(-t), (-1)\cos(-t) + 0\sin(-t)) \\ &= (\sin(-t), -\cos(-t)) \\ &= (-\sin t, -\cos t)\end{aligned}$$

Therefore, we obtain a much nicer expression for the same parametrized curve:

$$\alpha(t) = (t - \sin t, 1 - \cos t).$$

With this nicer expression of $\alpha(t)$, we obtain the derivative

$$\begin{aligned}\alpha'(t) = \frac{d\alpha}{dt} &= \frac{d}{dt}(t - \sin t, 1 - \cos t) \\ &= \left(\frac{d}{dt}(t - \sin t), \frac{d}{dt}(1 - \cos t)\right) \\ &= (1 - \cos t, \sin t).\end{aligned}$$

To find the critical points, we need to set $\alpha'(t) = 0$, which is equivalent to saying $1 - \cos t = 0$ and $\sin t = 0$. Note that, for all $n \in \mathbb{Z}$, $t = n\pi$ solve $\sin t = 0$; moreover, $t = n\pi$ also solves $\sin t = 0$ whenever n is even (i.e. $n = 2k$ for some $k \in \mathbb{Z}$). Therefore, all solutions to $\alpha'(t) = 0$ and hence the critical points of $\alpha(t)$ are $t = 2\pi k$ for any $k \in \mathbb{Z}$. □

b. Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Proof. In general, for all $t \in \mathbb{R}$ the half-angle trigonometric identity states

$$\sin\left(\frac{t}{2}\right) = \pm\sqrt{\frac{1 - \cos t}{2}}.$$

But $\sin(\frac{t}{2}) > 0$ for all $0 < \frac{t}{2} < \pi$ (or equivalently for all $0 < t < 2\pi$), which implies

$$\sin\left(\frac{t}{2}\right) = \sqrt{\frac{1 - \cos t}{2}}$$

for all $0 < t < 2\pi$. Therefore, we have

$$\begin{aligned}
L|_0^{2\pi} &= \int_0^{2\pi} |\alpha'(t)|\,dt \\
&= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\,dt \\
&= \int_0^{2\pi} \sqrt{(1 - 2\cos t + \cos^2 t) + \sin^2 t} \\
&= \int_0^{2\pi} \sqrt{1 - 2\cos t + (\cos^2 t + \sin^2 t)}\,dt \\
&= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\,dt \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos t}\,dt \\
&= \int_0^{2\pi} \sqrt{2(1-\cos t)}\,dt \\
&= 2\int_0^{2\pi} \sqrt{\frac{1-\cos t}{2}}\,dt \\
&= 2\int_0^{2\pi} \sin\left(\frac{t}{2}\right)\,dt \\
&= -4\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} \\
&= -4\left(\cos\left(\frac{2\pi}{2}\right) - \cos\left(\frac{0}{2}\right)\right) \\
&= -4(\cos\pi - \cos 0) \\
&= -4(-1-1) \\
&= 8,
\end{aligned}$$

the arc length of the cycloid corresponding to one complete rotation of the disk. □