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Solutions to assigned homework problems from *Differential Geometry of Curves and Surfaces* by Manfredo Perdigão do Carmo

Assignment 2 – pages 22-26: 1,4,7a,11,12,13,14

1-5.1. Consider for all s ∈ $\mathbb{R}$ the parametrized curve (helix)

$$\alpha(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b\frac{s}{c}\right),$$

where $c^2 = a^2 + b^2$.

a. Show that the parameter s is the arc length.

Proof. We have

$$\begin{aligned}\alpha'(s) = \frac{d\alpha}{ds} &= \frac{d}{ds}\left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b\frac{s}{c}\right)\\ &= \left(\frac{d}{ds}\left(a\cos\frac{s}{c}\right), \frac{d}{ds}\left(a\sin\left(\frac{s}{c}\right)\right), \frac{d}{ds}\left(b\frac{s}{c}\right)\right)\\ &= \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right),\end{aligned}$$

which means

$$\begin{aligned}|\alpha'(s)| &= \sqrt{\left(-\frac{a}{c}\sin\left(\frac{s}{c}\right)\right)^2 + \left(\frac{a}{c}\cos\left(\frac{s}{c}\right)\right)^2 + \left(\frac{b}{c}\right)^2}\\ &= \sqrt{\frac{a^2}{c^2}\left(\sin^2\left(\frac{s}{c}\right) + \cos^2\left(\frac{s}{c}\right)\right) + \frac{b^2}{c^2}}\\ &= \sqrt{\frac{a^2}{c^2} + \frac{b^2}{c^2}}\\ &= \sqrt{\frac{a^2+b^2}{c^2}}\\ &= \sqrt{\frac{a^2+b^2}{a^2+b^2}}\\ &= 1.\end{aligned}$$

In other words, $\alpha(s)$ has unit speed and hence is parametrized by arc langth. □

b. Determine the curvature and the torsion of α.

Proof. We have

$$\begin{aligned}\alpha''(s) = \frac{d\alpha'}{ds} &= \frac{d}{ds}\left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)\\ &= \left(\frac{d}{ds}\left(-\frac{a}{c}\sin\left(\frac{s}{c}\right)\right), \frac{d}{ds}\left(\frac{a}{c}\cos\left(\frac{s}{c}\right)\right), \frac{d}{ds}\left(\frac{b}{c}\right)\right)\\ &= \left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right).\end{aligned}$$

So we obtain the curvature

$$\begin{aligned}k &= |\alpha''(s)|\\ &= \sqrt{\left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right)\right)^2 + \left(-\frac{a}{c^2}\sin\left(\frac{s}{c}\right)\right)^2 + (0)^2}\\ &= \sqrt{\frac{a^2}{c^4}\left(\cos^2\left(\frac{s}{c}\right) + \sin^2\left(\frac{s}{c}\right)\right)}\\ &= \sqrt{\frac{a^2}{c^4}}\\ &= \frac{a}{c^2}\\ &= \frac{a}{a^2+b^2}.\end{aligned}$$

Next, we will compute the torsion of α. First, consider the tangent vector $t(s) = \alpha'(s) = (-\frac{a}{c}\sin(\frac{s}{c}), \frac{a}{c}\cos(\frac{s}{c}), \frac{b}{c})$. Then a unit normal vector is given by $n(s) = (\cos(\frac{s}{c}), \sin(\frac{s}{c}), 0)$, which is indeed orthogonal to $t(s)$ because

$$\begin{aligned} t(s)\cdot n(s) &= \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)\cdot\left(\cos\left(\frac{s}{c}\right), \sin\left(\frac{s}{c}\right), 0\right) \\ &= \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right)\right)\left(\cos\left(\frac{s}{c}\right)\right) + \left(\frac{a}{c}\cos\left(\frac{s}{c}\right)\right)\left(\sin\left(\frac{s}{c}\right)\right) + \left(\frac{b}{c}\right)(0) \\ &= -\frac{a}{c}\sin\left(\frac{s}{c}\right)\cos\left(\frac{s}{c}\right) + \frac{a}{c}\cos\left(\frac{s}{c}\right)\sin\left(\frac{s}{c}\right) \\ &= 0. \end{aligned}$$

Consequently, we obtain the binormal vector

$$\begin{aligned} b(s) &= t(s)\times n(s) \\ &= \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)\times\left(\cos\left(\frac{s}{c}\right), \sin\left(\frac{s}{c}\right), 0\right) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a}{c}\sin\left(\frac{s}{c}\right) & \frac{a}{c}\cos\left(\frac{s}{c}\right) & \frac{b}{c} \\ \cos\left(\frac{s}{c}\right) & \sin\left(\frac{s}{c}\right) & 0 \end{vmatrix} \\ &= \begin{vmatrix} \frac{a}{c}\cos(\frac{s}{c}) & \frac{b}{c} \\ \cos(\frac{s}{c}) & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -\frac{a}{c}\sin(\frac{s}{c}) & \frac{b}{c} \\ \cos(\frac{s}{c}) & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -\frac{a}{c}\sin(\frac{s}{c}) & \frac{a}{c}\cos(\frac{s}{c}) \\ \cos(\frac{s}{c}) & \sin(\frac{s}{c}) \end{vmatrix}\mathbf{k} \\ &= \left(\left(\frac{a}{c}\cos\left(\frac{s}{c}\right)\right)(0) - \left(\sin\left(\frac{s}{c}\right)\right)\left(\frac{b}{c}\right)\right)\mathbf{i} - \left(\left(-\frac{a}{c}\sin\left(\frac{s}{c}\right)\right)(0) - \left(\cos\left(\frac{s}{c}\right)\right)\left(\frac{b}{c}\right)\right)\mathbf{j} \\ &\quad + \left(\left(-\frac{a}{c}\sin\left(\frac{s}{c}\right)\right)\left(\sin\left(\frac{s}{c}\right)\right) - \left(\cos\left(\frac{s}{c}\right)\right)\left(\frac{a}{c}\cos\left(\frac{s}{c}\right)\right)\right)\mathbf{k} \\ &= -\frac{b}{c}\sin\left(\frac{s}{c}\right)\mathbf{i} + \frac{b}{c}\cos\left(\frac{s}{c}\right)\mathbf{j} - \frac{a}{c}\left(\sin^2\left(\frac{s}{c}\right) + \cos^2\left(\frac{s}{c}\right)\right)\mathbf{k} \\ &= -\frac{b}{c}\sin\left(\frac{s}{c}\right)\mathbf{i} + \frac{b}{c}\cos\left(\frac{s}{c}\right)\mathbf{j} - \frac{a}{c}\mathbf{k} \\ &= \left(-\frac{b}{c}\sin\left(\frac{s}{c}\right), \frac{b}{c}\cos\left(\frac{s}{c}\right), -\frac{a}{c}\right) \end{aligned}$$

and its derivative

$$\begin{aligned} b'(s) &= \frac{d}{ds}\left(-\frac{b}{c}\sin\left(\frac{s}{c}\right), \frac{b}{c}\cos\left(\frac{s}{c}\right), -\frac{a}{c}\right) \\ &= \left(\frac{d}{ds}\left(-\frac{b}{c}\sin\left(\frac{s}{c}\right)\right), \frac{d}{ds}\left(\frac{b}{c}\cos\left(\frac{s}{c}\right)\right), \frac{d}{ds}\left(-\frac{a}{c}\right)\right) \\ &= \left(-\frac{b}{c^2}\cos\left(\frac{s}{c}\right), -\frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right). \end{aligned}$$

Hence, we obtain the torsion

$$\begin{aligned} \tau(s) &= b'(s)\cdot n(s) \\ &= \left(-\frac{b}{c^2}\cos\left(\frac{s}{c}\right), -\frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right)\cdot\left(\cos\left(\frac{s}{c}\right), \sin\left(\frac{s}{c}\right), 0\right) \\ &= \left(-\frac{b}{c^2}\cos\left(\frac{s}{c}\right)\right)\left(\cos\left(\frac{s}{c}\right)\right) + \left(-\frac{b}{c^2}\sin\left(\frac{s}{c}\right)\right)\left(\sin\left(\frac{s}{c}\right)\right) + (0)(0) \\ &= -\frac{b}{c^2}\left(\cos^2\left(\frac{s}{c}\right) + \sin^2\left(\frac{s}{c}\right)\right) \\ &= -\frac{b}{c^2} \\ &= -\frac{b}{a^2+b^2}, \end{aligned}$$

as desired. □

c. Determine the osculating plane of α.

Proof. Let $p = (x, y, z) \in \mathbb{R}^3$ be some point that does not lie on the trace of α. Then $p - \alpha(s) = \lambda n(s)$ for some scalar $\lambda \in \mathbb{R}$. We recall that the *osculating plane* at s is a plane that is determined by the tangent vector $t(s) = \alpha'(s)$ and the normal vector $n(s)$ (c.f. do Carmo, page 17). As a result, we have that p lies in the osculating plane at s if and only if we have

$$\begin{aligned} b(s)\cdot(p-\alpha(s)) &= b(s)\cdot\lambda n(s)\\ &= \lambda b(s)\cdot n(s)\\ &= 0. \end{aligned}$$

But at the same time we also have

$$\begin{aligned} b(s)\cdot(p-\alpha(s)) &= \left(-\frac{b}{c}\sin\left(\frac{s}{c}\right), \frac{b}{c}\cos\left(\frac{s}{c}\right), -\frac{a}{c}\right)\cdot\left((x,y,z)-\left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b\frac{s}{c}\right)\right)\\ &= \left(-\frac{b}{c}\sin\left(\frac{s}{c}\right), \frac{b}{c}\cos\left(\frac{s}{c}\right), -\frac{a}{c}\right)\cdot\left(x-a\cos\left(\frac{s}{c}\right), y-a\sin\left(\frac{s}{c}\right), z-b\frac{s}{c}\right)\\ &= \left(-\frac{b}{c}\sin\left(\frac{s}{c}\right)\right)\left(x-a\cos\left(\frac{s}{c}\right)\right)+\left(\frac{b}{c}\cos\left(\frac{s}{c}\right)\right)\left(y-a\sin\left(\frac{s}{c}\right)\right)+\left(-\frac{a}{c}\right)\left(z-b\frac{s}{c}\right)\\ &= \left(-\frac{b}{c}\sin\left(\frac{s}{c}\right)x+\frac{ab}{c}\sin\left(\frac{s}{c}\right)\cos\left(\frac{s}{c}\right)\right)+\left(\frac{b}{c}\cos\left(\frac{s}{c}\right)y-\frac{ab}{c}\cos\left(\frac{s}{c}\right)\sin\left(\frac{s}{c}\right)\right)+\left(-\frac{a}{c}z+\frac{ab}{c}\frac{s}{c}\right)\\ &= -\frac{b}{c}\sin\left(\frac{s}{c}\right)x+\frac{b}{c}\cos\left(\frac{s}{c}\right)y-\frac{a}{c}z+\frac{ab}{c}\frac{s}{c}. \end{aligned}$$

We equate the above two expressions to conclude that

$$-\frac{b}{c}\sin\left(\frac{s}{c}\right)x+\frac{b}{c}\cos\left(\frac{s}{c}\right)y-\frac{a}{c}z+\frac{ab}{c}\frac{s}{c}=0,$$

or equivalently

$$b\sin\left(\frac{s}{c}\right)x-b\cos\left(\frac{s}{c}\right)y+az=ab\frac{s}{c},$$

is the equation for the osculating plane of α. □

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z-axis under a constant angle equal to $\frac{\pi}{2}$.

Proof. Let θ be the angle between $n(s)$ and the z-axis; we will show that $\theta=\frac{\pi}{2}$. Recall the unit normal vector $n(s)=(\cos(\frac{s}{c}), \sin(\frac{s}{c}), 0)$, which of course has magnitude $|n(s)|=\sqrt{(\cos(\frac{s}{c}))^2+(\sin(\frac{s}{c}))^2+(0)^2}=1$. Also consider the vectors $(0,0,1)$ and $(0,0,-1)$, the upward and downward pointing unit vectors in the z-axis, respectively; they both have the same magnitude $|(0,0,\pm1)|=\sqrt{(0)^2+(0)^2+(\pm1)^2}=1$. So we have

$$\begin{aligned} \cos\theta &= \frac{n(s)\cdot(0,0,\pm1)}{|n(s)||(0,0,\pm1)|}\\ &= \frac{(\cos(\frac{s}{c}), \sin(\frac{s}{c}), 0)\cdot(0,0,1)}{(1)(1)}\\ &= \left(\cos\left(\frac{s}{c}\right)\right)(0)+\left(\sin\left(\frac{s}{c}\right)\right)(0)+(0)(1)\\ &= 0, \end{aligned}$$

and so $\theta=\cos^{-1}(0)=\frac{\pi}{2}$. □

e. Show that the tangent lines to α make a constant angle with the z-axis.

Proof. Let θ be the angle between $\alpha'(s)$ and the z-axis (to clarify, this is *not* the same angle θ from part d). Also consider as in part (d) the vectors $(0,0,\pm1)$ in the z-axis. So we have

$$\begin{aligned} \cos\theta &= \frac{\alpha'(s)\cdot(0,0,\pm1)}{|\alpha'(s)||(0,0,\pm1)|}\\ &= \frac{(-\frac{a}{c}\sin(\frac{s}{c}), \frac{a}{c}\cos(\frac{s}{c}), \frac{b}{c})\cdot(0,0,\pm1)}{(1)(1)}\\ &= \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right)\right)(0)+\left(\frac{a}{c}\cos\left(\frac{s}{c}\right)\right)(0)+\left(\frac{b}{c}\right)(\pm1)\\ &= \pm\frac{b}{c}, \end{aligned}$$

which implies the angles $\theta=\cos^{-1}(\frac{b}{c})$ and $\theta=\cos^{-1}(-\frac{b}{c})$. Since neither of these two values depends on s, we conclude that θ is a constant angle (that only depends on the choice of one of the two unit vectors $(0,0,1)$ or $(0,0,-1)$ in the z-axis). □

1-5.4. Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.

Proof. Let p be a fixed point and $n(s)$ a unit normal vector of the parametrized curve; that is, $\alpha'(s) \cdot n(s) = 0$. Since $n(s)$ passes through p, we have $\alpha(s) - p = \lambda n(s)$ for some scalar $\lambda \in \mathbb{R}$. So we have

$$\begin{aligned}
\frac{d}{ds}(|\alpha(s) - p|^2) &= \frac{d}{ds}((\alpha(s) - p) \cdot (\alpha(s) - p)) \\
&= \frac{d}{ds}(\alpha(s) - p) \cdot (\alpha(s) - p) + (\alpha(s) - p) \cdot \frac{d}{ds}(\alpha(s) - p) \\
&= 2\frac{d}{ds}(\alpha(s) - p) \cdot (\alpha(s) - p) \\
&= 2\alpha'(s) \cdot (\alpha(s) - p) \\
&= 2\alpha'(s) \cdot \lambda n(s) \\
&= 2\lambda\alpha'(s) \cdot n(s) \\
&= 0.
\end{aligned}$$

Hence, $|\alpha(s) - p|^2$ is constant, which means $|\alpha(s) - p|$ is constant. In other words, for all $s \in I$ the distance between $\alpha(s)$ and p is the same, which implies that the trace of $\alpha(s)$ is contained in a circle of center p. □

1-5.7. Let $\alpha : I \to \mathbb{R}^2$ be a regular parametrized plane curve (arbitrary parameter), and define $n = n(t)$ and $k = k(t)$ as in Remark 1 (c.f. do Carmo, page 21). Assume that $k(t) \neq 0$, $t \in I$. In this situation, the curve

$$\beta(t) = \alpha(t) + \frac{1}{k(t)}n(t)$$

is called the *evolute* of α.

a. Show that the tangent at t of the evolute of α is the normal to α at t.

Proof. Let $\alpha'(t)$ be the tangent vector to α and $n(t)$ the normal vector to α. As tangent and normal vectors are always perpendicular to each other, we have $n(t) \cdot \alpha'(t) = 0$. First, we obtain the derivative of the evolute

$$\begin{aligned}
\beta'(t) = \frac{d\beta}{dt} &= \frac{d}{dt}\left(\alpha(t) + \frac{1}{k(t)}n(t)\right) \\
&= \frac{d}{dt}\alpha(t) + \frac{d}{dt}\left(\frac{n(t)}{k(t)}\right) \\
&= \alpha'(t) + \frac{k(t)n'(t) - n(t)k'(t)}{(k(t))^2} \\
&= \alpha'(t) + \frac{1}{k(t)}n'(t) - \frac{k'(t)}{(k(t))^2}n(t).
\end{aligned}$$

Since α is a plane curve (i.e. $\alpha(t) = (x(t), y(t)) \in \mathbb{R}^2$), it follows that we have zero torsion, or $\tau = 0$ (c.f. do Carmo, page 18). So the second of the three Frenet formulas $n' = -kt - \tau b$ (c.f. do Carmo, page 19) reduces to $n' = -kt$ for plane curves. Moreover, if necessary we can reparametrize the curve by writing $t = t(s)$ and its inverse $s = s(t)$, which results in, for example, $\alpha(t) = \alpha(t(s)) = \alpha(s)$ or $k(s) = k(s(t)) = k(t)$, though this is considered an abuse of notation. So we can identify the tangent vector $t(s)$ with $\alpha'(s)$ and therefore write $n'(s) = -k(s)\alpha'(s)$, or equivalently $\frac{1}{k(s)}n'(s) = -\alpha'(s)$. Therefore, we have

$$\begin{aligned}
\beta'(t) \cdot \alpha'(t) &= \left(\alpha'(t) + \frac{1}{k(t)}n'(t) - \frac{k'(t)}{(k(t))^2}n(t)\right) \cdot \alpha'(t) \\
&= \alpha'(t) \cdot \alpha'(t) + \frac{1}{k(t)}n'(t) \cdot \alpha'(t) - \frac{k'(t)}{(k(t))^2}n(t) \cdot \alpha'(t) \\
&= \alpha'(t) \cdot \alpha'(t) + \frac{1}{k(t)}n'(t) \cdot \alpha'(t) - \frac{k'(t)}{(k(t))^2}n(t) \cdot \alpha'(t) \\
&= \alpha'(t) \cdot \alpha'(t) + \frac{1}{k(t)}n'(t) \cdot \alpha'(t) \\
&= \alpha'(t) \cdot \alpha'(t) + \frac{1}{k(s)}n'(s) \cdot \alpha'(t) \\
&= \alpha'(t) \cdot \alpha'(t) + \alpha'(s) \cdot \alpha'(t) \\
&= \alpha'(t) \cdot \alpha'(t) - \alpha'(t) \cdot \alpha'(t) \\
&= 0,
\end{aligned}$$

which establishes that the evolute $\beta(t)$ is normal to α at t. □

1-5.11. One often gives a plane curve in polar coordinates by $\rho = \rho(\theta)$, $a \leq \theta \leq b$.

a. Show that the arc length is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta,$$

where the prime denotes the derivative relative to θ.

Proof. Let $L|_a^b$ denote the arc length of the plane curve $\alpha(\theta) = (x(\theta), y(\theta))$. Let $x(\theta) = \rho(\theta)\cos\theta$ and $y(\theta) = \rho(\theta)\sin\theta$. Then

$$\begin{aligned} x'(\theta) = \frac{dx}{d\theta} &= \frac{d}{d\theta}(\rho(\theta)\cos\theta) \\ &= \rho'(\theta)\cos\theta - \rho(\theta)\sin\theta \end{aligned}$$

and

$$\begin{aligned} y'(\theta) = \frac{dy}{d\theta} &= \frac{d}{d\theta}(\rho(\theta)\sin\theta) \\ &= \rho'(\theta)\sin\theta + \rho(\theta)\cos\theta. \end{aligned}$$

So we have

$$\begin{aligned} (x'(\theta))^2 + (y'(\theta))^2 &= (\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta)^2 + (\rho'(\theta)\sin\theta + \rho(\theta)\cos\theta)^2 \\ &= ((\rho'(\theta))^2\cos^2\theta - 2\rho(\theta)\rho'(\theta)\cos\theta\sin\theta + (\rho(\theta))^2\sin^2\theta) \\ &\quad + ((\rho'(\theta))^2\sin^2\theta + 2\rho(\theta)\rho'(\theta)\cos\theta\sin\theta + (\rho(\theta))^2\cos^2\theta) \\ &= (\rho'(\theta))^2 + (\rho(\theta))^2, \end{aligned}$$

and so

$$\begin{aligned} L|_a^b &= \int_a^b \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta \\ &= \int_a^b \sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2}\, d\theta, \end{aligned}$$

as desired. □

b. Show that the curvature is

$$k(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{((\rho')^2 + \rho^2)^{\frac{3}{2}}}.$$

Proof. We will use the formula for curvature in Exercise 1-5.12 part d below (replacing t with θ) in Cartesian coordinates, which is

$$k(\theta) = \frac{x'(\theta)y''(\theta) - x''(\theta)y'(\theta)}{((x'(\theta))^2 + (y'(\theta))^2)^{\frac{3}{2}}},$$

and we would like to convert this into polar coordinates. We already have from our proof of part a that

$$(x'(\theta))^2 + (y'(\theta))^2 = (\rho'(\theta))^2 + (\rho(\theta))^2.$$

So it suffices to show that

$$x'(\theta)y''(\theta) - x''(\theta)y'(\theta) = 2(\rho'(\theta))^2 + \rho(\theta)\rho''(\theta) + (\rho(\theta))^2.$$

To this end, we recall the first derivatives $x'(\theta) = \rho'(\theta)\cos\theta - \rho(\theta)\sin\theta$ and $y'(\theta) = \rho'(\theta)\sin\theta + \rho(\theta)\cos\theta$ to obtain the second derivatives

$$\begin{aligned} x''(\theta) = \frac{dx'}{d\theta} &= \frac{d}{d\theta}(\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta) \\ &= \frac{d}{d\theta}(\rho'(\theta)\cos\theta) - \frac{d}{d\theta}(\rho(\theta)\sin\theta) \\ &= (\rho''(\theta)\cos\theta - \rho'(\theta)\sin\theta) - (\rho'(\theta)\sin\theta + \rho(\theta)\cos\theta) \\ &= \rho''(\theta)\cos\theta - 2\rho'(\theta)\sin\theta - \rho(\theta)\cos\theta \end{aligned}$$

and

$$\begin{aligned} y''(\theta) = \frac{dy'}{d\theta} &= \frac{d}{d\theta}(\rho'(\theta)\sin\theta + \rho(\theta)\cos\theta) \\ &= \frac{d}{d\theta}(\rho'(\theta)\sin\theta) + \frac{d}{d\theta}(\rho(\theta)\cos\theta) \\ &= (\rho''(\theta)\sin\theta + \rho'(\theta)\cos\theta) + (\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta) \\ &= \rho''(\theta)\sin\theta + 2\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta. \end{aligned}$$

So we have

$$\begin{aligned}
x'(\theta)y''(\theta) - x''(\theta)y'(\theta) &= (\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta)(\rho''(\theta)\sin\theta + 2\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta) \\
&\quad - (\rho''(\theta)\cos\theta - 2\rho'(\theta)\sin\theta - \rho(\theta)\cos\theta)(\rho'(\theta)\sin\theta + \rho(\theta)\cos\theta) \\
&= \rho'(\theta)\rho''(\theta)\sin\theta\cos\theta + 2(\rho'(\theta))^2\cos^2\theta - \rho(\theta)\rho'(\theta)\sin\theta\cos\theta \\
&\quad - \rho(\theta)\rho''(\theta)\sin^2\theta - 2\rho(\theta)\rho'(\theta)\sin\theta\cos\theta + (\rho(\theta))^2\sin^2\theta \\
&\quad - \rho'(\theta)\rho''(\theta)\sin\theta\cos\theta + 2(\rho'(\theta))^2\sin^2\theta + \rho(\theta)\rho'(\theta)\sin\theta\cos\theta \\
&\quad - \rho(\theta)\rho''(\theta)\cos^2\theta + 2\rho(\theta)\rho''(\theta)\sin\theta\cos\theta + (\rho(\theta))^2\cos^2\theta \\
&= 2(\rho'(\theta))^2(\cos^2\theta + \sin^2\theta) - \rho(\theta)\rho''(\theta)(\sin^2\theta + \cos^2\theta) + (\rho(\theta))^2(\sin^2\theta + \cos^2\theta) \\
&= 2(\rho'(\theta))^2 - \rho(\theta)\rho''(\theta) + (\rho(\theta))^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
k(\theta) &= \frac{x'(\theta)y''(\theta) - x''(\theta)y'(\theta)}{((x'(\theta))^2 + (y'(\theta))^2)^{\frac{3}{2}}} \\
&= \frac{2(\rho'(\theta))^2 - \rho(\theta)\rho''(\theta) + (\rho(\theta))^2}{((\rho'(\theta))^2 + (\rho(\theta))^2)^{\frac{3}{2}}},
\end{aligned}$$

as desired. □

1-5.12. Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve (not necessarily by arc length) and let $\beta : J \to \mathbb{R}^3$ be a reparametrization of $\alpha(I)$ be the arc length $s = s(t)$, measured from $t_0 \in I$ (see Remark 2; c.f. do Carmo, pages 21–22). Let $t = t(s)$ be the inverse function of s and set $\frac{d\alpha}{dt} = \alpha'$, $\frac{d^2\alpha}{dt^2} = \alpha''$, etc. Prove that:

a. $\frac{dt}{ds} = \frac{1}{|\alpha'|}$ and $\frac{d^2t}{ds^2} = -\frac{\alpha'\cdot\alpha''}{|\alpha'|^4}$.

Proof. Since we have $t = t(s)$, we can reparametrize α by writing $\beta(s) = \alpha(t(s))$. So β is parametrized by arc length, which means $|\beta'(s)| = 1$. Thus,

$$\begin{aligned}
1 &= |b'(s)|^2 \\
&= \beta'(s)\cdot\beta'(s) \\
&= \frac{d\beta}{ds}\cdot\frac{d\beta}{ds} \\
&= \frac{d\alpha}{ds}(t(s))\cdot\frac{d\alpha}{ds}(t(s)) \\
&= \frac{d\alpha}{dt}\frac{dt}{ds}\cdot\frac{d\alpha}{dt}\frac{dt}{ds} \\
&= \alpha'(t)\frac{dt}{ds}\cdot\alpha'(t)\frac{dt}{ds} \\
&= \left(\frac{dt}{ds}\right)^2\alpha'(t)\cdot\alpha'(t) \\
&= \left(\frac{dt}{ds}\right)^2|\alpha'(t)|^2,
\end{aligned}$$

from which upon taking square roots of both sides we conclude $1 = \frac{dt}{ds}|\alpha'(t)|$, or $\frac{dt}{ds} = \frac{1}{|\alpha'(t)|}$. To obtain the expression of the second dreivative $\frac{d^2t}{ds^2}$, first we recall our already-established expression

$$1 = \left(\frac{dt}{ds}\right)^2\alpha'(t)\cdot\alpha'(t),$$

from which we take the derivative in s of both sides to obtain

$$0 = \frac{d}{ds}\left(\left(\frac{dt}{ds}\right)^2\alpha'(t)\cdot\alpha'(t)\right).$$

But then we also have

$$\begin{aligned}
\frac{d}{ds}\left(\left(\frac{dt}{ds}\right)^2 \alpha'(t)\cdot\alpha'(t)\right) &= \frac{d}{ds}\left(\left(\frac{dt}{ds}\right)^2\right)\alpha'(t)\cdot\alpha'(t) + \left(\frac{dt}{ds}\right)^2 \frac{d}{ds}(\alpha'(t)\cdot\alpha'(t)) \\
&= \left(2\frac{dt}{ds}\frac{d^2t}{ds^2}\right)|\alpha'(t)|^2 + \left(\frac{dt}{ds}\right)^2\left(\frac{d}{ds}(\alpha'(t(s)))\cdot\alpha'(t) + \alpha'(t)\cdot\frac{d}{ds}\alpha'(t(s))\right) \\
&= 2\frac{dt}{ds}\frac{d^2t}{ds^2}|\alpha'(t)|^2 + \left(\frac{dt}{ds}\right)^2\left(2\frac{d}{ds}(\alpha'(t(s)))\cdot\alpha'(t)\right) \\
&= 2\left(\frac{dt}{ds}\frac{d^2t}{ds^2}|\alpha'(t)|^2 + \left(\frac{dt}{ds}\right)^2\frac{d\alpha'}{dt}\frac{dt}{ds}\cdot\alpha'(t)\right) \\
&= 2\left(\frac{1}{|\alpha'(t)|}\frac{d^2t}{ds^2}|\alpha'(t)|^2 + \left(\frac{1}{|\alpha'(t)|}\right)^2\alpha''(t)\frac{1}{|\alpha'(t)|}\cdot\alpha'(t)\right) \\
&= 2\left(|\alpha'(t)|\frac{d^2t}{ds^2} + \frac{1}{|\alpha'(t)|^3}\alpha''(t)\cdot\alpha'(t)\right).
\end{aligned}$$

Equating the last two expressions together, we conclude that

$$|\alpha'(t)|\frac{d^2t}{ds^2} + \frac{1}{|\alpha'(t)|^3}\alpha''(t)\cdot\alpha'(t) = 0,$$

or $\frac{d^2t}{ds^2} = -\frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^4}$. □

b. The curvature of α at $t \in I$ is

$$k(t) := \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.$$

Proof. We recall the vector identity $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$ (c.f. do Carmo, page 14). Substituting the values $u = \alpha'(t)$, $v = \alpha''(t)$, $w = \alpha'(t)$, this vector identity becomes

$$(\alpha'(t) \times \alpha''(t)) \times \alpha'(t) = (\alpha'(t)\cdot\alpha'(t))\alpha''(t) - (\alpha'(t)\cdot\alpha''(t))\alpha'(t).$$

So we have

$$\begin{aligned}
\beta''(s) = \frac{d}{ds}\beta'(s) = \frac{d}{ds}\left(\frac{d}{ds}\alpha(t(s))\right) &= \frac{d}{ds}\left(\frac{d\alpha}{dt}\frac{dt}{ds}\right) \\
&= \frac{d}{ds}\left(\frac{d\alpha}{dt}\right)\frac{dt}{ds} + \frac{d\alpha}{dt}\frac{d}{ds}\left(\frac{dt}{ds}\right) \\
&= \left(\frac{d^2\alpha}{dt^2}\frac{dt}{ds}\right)\frac{dt}{ds} + \frac{d\alpha}{dt}\frac{d^2t}{ds^2} \\
&= \frac{d^2\alpha}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{d\alpha}{dt}\frac{d^2t}{ds^2} \\
&= \alpha''(t)\left(\frac{1}{|\alpha'(t)|}\right)^2 + \alpha'(t)\left(-\frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^4}\right) \\
&= \frac{\alpha''(t)}{|\alpha'(t)|^2} - \frac{(\alpha'(t)\cdot\alpha''(t))\alpha'(t)}{|\alpha'(t)|^4} \\
&= \frac{\alpha'(t)\cdot\alpha'(t)}{|\alpha'(t)|^2}\frac{\alpha''(t)}{|\alpha'(t)|^2} - \frac{(\alpha'(t)\cdot\alpha''(t))\alpha'(t)}{|\alpha'(t)|^4} \\
&= \frac{(\alpha'(t)\cdot\alpha'(t))\alpha''(t) - (\alpha'(t)\cdot\alpha''(t))\alpha'(t)}{|\alpha'(t)|^4} \\
&= \frac{(\alpha'(t)\times\alpha''(t))\times\alpha'(t)}{|\alpha'(t)|^4},
\end{aligned}$$

Also, we recall that if u, v are perpendicular vectors (i.e. the angle between u, v is a right angle, or $\frac{\pi}{2}$ radians), then $|u \times v| = |u||v| \sin\frac{\pi}{2} = |u||v|$. We also recall that the cross product $\alpha'(t) \times \alpha''(t)$ is perpendicular to both vectors $\alpha'(t)$ and $\alpha''(t)$. Thus, with our substitutions $u = \alpha'(t) \times \alpha''(t)$ and $v = \alpha'(t)$ (not the same substitutions to our u, v from earlier), we have

$$\begin{aligned}
|(\alpha'(t)\times\alpha''(t))\times\alpha'(t)| &= |\alpha'(t)\times\alpha''(t)||\alpha'(t)|\sin\frac{\pi}{2} \\
&= |\alpha'(t)\times\alpha''(t)||\alpha'(t)|.
\end{aligned}$$

Finally, the definition on page 16 of do Carmo gives us $k(t) = |\alpha''(t)|$. Therefore, we have

$$\begin{aligned}
k(t) = k(t(s)) = k(s) &= |\beta''(s)| \\
&= \left|\frac{(\alpha'(t)\times\alpha''(t))\times\alpha'(t)}{|\alpha'(t)|^4}\right| \\
&= \frac{|(\alpha'(t)\times\alpha''(t))\times\alpha'(t)|}{|\alpha'(t)|^4} \\
&= \frac{|\alpha'(t)\times\alpha''(t)||\alpha'(t)|}{|\alpha'(t)|^4} \\
&= \frac{|\alpha'(t)\times\alpha''(t)|}{|\alpha'(t)|^3},
\end{aligned}$$

as desired. □

c. The torsion of α at $t \in I$ is

$$\tau(t) := -\frac{(\alpha'\times\alpha'')\cdot\alpha'''}{|\alpha'\times\alpha''|^2}.$$

Proof. Since a curve like $\beta(s)$ is parameterized by arc length, i.e. $|\beta'(s)| = 1$, obviously $|\beta'(s)|$ is constant in s. By Exercise 1-2.5 (replace the $\alpha(t)$ in Exercise 1-2.5 by $\beta'(s)$), we conclude that $\beta''(s)$ is orthogonal to $\beta'(s)$. This means $\beta''(s)$ lies in the same direction of the unit normal vector $n(s)$ and is therefore a scalar multiple of $n(s)$; in particular, we can write $n(s) = \frac{\beta''(s)}{|\beta''(s)|}$. An immediate consequence of this is that, since any cross product of two same vectors is zero (i.e. $n(s)\times n(s) = 0$), we have

$$\begin{aligned}
t'(s)\times n(s) &= \beta''(s)\times n(s) \\
&= |\beta''(s)|n(s)\times n(s) \\
&= 0.
\end{aligned}$$

With that said, we recall that the torsion is expressed as the equation $b' = \tau n$, or equivalently $\tau = b'\cdot n$. And note that the cross product $\beta'(s)\times\beta''(s)$ is a vector that is perpendicular to both $\beta'(s)$ and $\beta''(s)$; in particular, $\beta'(s)\times\beta''(s)$ is orthogonal to $\beta''(s)$, which is saying $(\beta'(s)\times\beta''(s))\cdot\beta''(s) = 0$. Thus, we have

$$\begin{aligned}
\tau(t) = \tau(t(s)) = \tau(s) = b'(s)\cdot n(s) &= (t(s)\times n(s))'\cdot n(s) \\
&= (t'(s)\times n(s) + t(s)\times n'(s))\cdot n(s) \\
&= (t(s)\times n'(s))\cdot n(s) \\
&= \left(\beta'(s)\times\frac{d}{ds}\left(\frac{\beta''(s)}{|\beta''(s)|}\right)\right)\cdot\frac{\beta''(s)}{|\beta''(s)|} \\
&= \left(\beta'(s)\times\frac{|\beta''(s)|\beta'''(s)-\beta''(s)\frac{d}{ds}|\beta''(s)|}{|\beta''(s)|^2}\right)\cdot\frac{\beta''(s)}{|\beta''(s)|} \\
&= \left(\beta'(s)\times\left(\frac{\beta'''(s)}{|\beta''(s)|}-\frac{\beta''(s)\frac{d}{ds}|\beta''(s)|}{|\beta''(s)|^2}\right)\right)\cdot\frac{\beta''(s)}{|\beta''(s)|} \\
&= \frac{1}{|\beta''(s)|^2}(\beta'(s)\times\beta'''(s))\cdot\beta''(s) - \frac{\frac{d}{ds}|\beta''(s)|}{|\beta''(s)|^3}(\beta'(s)\times\beta''(s))\cdot\beta''(s) \\
&= \frac{1}{|\beta''(s)|^2}(\beta'(s)\times\beta'''(s))\cdot\beta''(s).
\end{aligned}$$

It is now a good idea to write separately the expressions of $\beta'(s)$, $\beta''(s)$, $\beta'''(s)$, $|\beta''(s)|^2$. First,

$$\begin{aligned}
\beta'(s) = \frac{d\beta}{ds} = \frac{d}{ds}\alpha(t(s)) &= \frac{d\alpha}{dt}\frac{dt}{ds} \\
&= \alpha'(t)\frac{1}{|\alpha'(t)|} \\
&= \frac{\alpha'(t)}{|\alpha'(t)|}.
\end{aligned}$$

Second,

$$
\begin{aligned}
\beta''(s) &= \frac{d}{ds}\beta'(s) = \frac{d}{ds}\left(\frac{d}{ds}\alpha(t(s))\right) = \frac{d}{ds}\left(\frac{d\alpha}{dt}\frac{dt}{ds}\right) \\
&= \frac{d}{ds}\left(\frac{d\alpha}{dt}\right)\frac{dt}{ds} + \frac{d\alpha}{dt}\frac{d}{ds}\left(\frac{dt}{ds}\right) \\
&= \left(\frac{d^2\alpha}{dt^2}\frac{dt}{ds}\right)\frac{dt}{ds} + \frac{d\alpha}{dt}\frac{d^2t}{ds^2} \\
&= \frac{d^2\alpha}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{d\alpha}{dt}\frac{d^2t}{ds^2} \\
&= \alpha''(t)\left(\frac{1}{|\alpha'(t)|}\right)^2 + \alpha'(t)\left(-\frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^4}\right) \\
&= \frac{\alpha''(t)}{|\alpha'(t)|^2} - \frac{(\alpha'(t)\cdot\alpha''(t))\alpha'(t)}{|\alpha'(t)|^4} \\
&= \frac{\alpha'(t)\cdot\alpha'(t)}{|\alpha'(t)|^2}\frac{\alpha''(t)}{|\alpha'(t)|^2} - \frac{(\alpha'(t)\cdot\alpha''(t))\alpha'(t)}{|\alpha'(t)|^4} \\
&= \frac{(\alpha'(t)\cdot\alpha'(t))\alpha''(t) - (\alpha'(t)\cdot\alpha''(t))\alpha'(t)}{|\alpha'(t)|^4} \\
&= \frac{(\alpha'(t)\times\alpha''(t))\cdot\alpha'(t)}{|\alpha'(t)|^4}.
\end{aligned}
$$

Third,

$$
\begin{aligned}
\beta'''(s) &= \frac{d}{ds}\beta''(s) = \frac{d}{ds}\left(\frac{d^2\alpha}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{d\alpha}{dt}\frac{d^2t}{ds^2}\right) \\
&= \frac{d}{ds}\left(\frac{d^2\alpha}{dt^2}\left(\frac{dt}{ds}\right)^2\right) + \frac{d}{ds}\left(\frac{d\alpha}{dt}\frac{d^2t}{ds^2}\right) \\
&= \left(\frac{d}{ds}\left(\frac{d^2\alpha}{dt^2}\right)\left(\frac{dt}{ds}\right)^2 + \left(\frac{d^2\alpha}{dt^2}\right)\frac{d}{ds}\left(\left(\frac{dt}{ds}\right)^2\right)\right) + \left(\frac{d}{ds}\left(\frac{d\alpha}{dt}\right)\left(\frac{d^2t}{ds^2}\right) + \left(\frac{d\alpha}{dt}\right)\frac{d}{ds}\left(\frac{d^2t}{ds^2}\right)\right) \\
&= \left(\left(\frac{d^3\alpha}{dt^3}\frac{dt}{ds}\right)\left(\frac{dt}{ds}\right)^2 + \left(\frac{d^2\alpha}{dt^2}\right)\left(2\frac{dt}{ds}\frac{d^2t}{ds^2}\right)\right) + \left(\left(\frac{d^2\alpha}{dt^2}\frac{dt}{ds}\right)\left(\frac{d^2t}{ds^2}\right) + \left(\frac{d\alpha}{dt}\right)\left(\frac{d^3t}{ds^3}\right)\right) \\
&= \frac{d^3\alpha}{dt^3}\left(\frac{dt}{ds}\right)^3 + 3\frac{d^2\alpha}{dt^2}\frac{dt}{ds}\frac{d^2t}{ds^2} + \frac{d\alpha}{dt}\frac{d^3t}{ds^3} \\
&= \alpha'''(t)\left(\frac{1}{|\alpha'(t)|}\right)^3 + 3\alpha''(t)\left(\frac{1}{|\alpha'(t)|}\right)\left(-\frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^4}\right) + \alpha'(t)\frac{d^3t}{ds^3} \\
&= \frac{1}{|\alpha'(t)|^3}\alpha'''(t) - 3\frac{\alpha'(t)\dot{\alpha}''(t)}{|\alpha'(t)|^5}\alpha''(t) + \frac{d^3t}{ds^3}\alpha'(t).
\end{aligned}
$$

A side note from the second and third items is that we can obtain the cross product

$$
\begin{aligned}
\beta'(s)\times\beta'''(s) &= \frac{\alpha'(t)}{|\alpha'(t)|}\times\left(\frac{1}{|\alpha'(t)|^3}\alpha'''(t) - 3\frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^5}\alpha''(t) + \frac{d^3t}{ds^3}\alpha'(t)\right) \\
&= \frac{1}{|\alpha'(t)|^4}\alpha'(t)\times\alpha'''(t) - \frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^6}\alpha'(t)\times\alpha''(t) + \frac{1}{|\alpha'(t)|}\frac{d^3t}{ds^3}\alpha'(t)\times\alpha'(t) \\
&= \frac{1}{|\alpha'(t)|^4}\alpha'(t)\times\alpha'''(t) - \frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^6}\alpha'(t)\times\alpha''(t)
\end{aligned}
$$

and consequently, as cross product vectors are perpendicular to their input vectors into the cross product, we have $(\alpha'(t)\times\alpha'''(t))\cdot\alpha'(t) = 0$, $(\alpha'(t)\times\alpha''(t))\cdot\alpha'(t) = 0$, and $(\alpha'(t)\times\alpha''(t))\cdot\alpha'(t) = 0$, and so we also get the dot product

$$
\begin{aligned}
(\beta'(s)\times\beta'''(s))\cdot\beta''(s) &= \left(\frac{1}{|\alpha'(t)|^4}\alpha'(t)\times\alpha'''(t) - \frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^6}\alpha'(t)\times\alpha''(t)\right)\cdot\left(\frac{\alpha''(t)}{|\alpha'(t)|^2} - \frac{(\alpha'(t)\cdot\alpha''(t))\alpha'(t)}{|\alpha'(t)|^4}\right) \\
&= \frac{1}{|\alpha'(t)|^6}(\alpha'(t)\times\alpha'''(t))\cdot\alpha''(t) - \frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^8}(\alpha'(t)\times\alpha'''(t))\cdot\alpha'(t) \\
&\quad - \frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^8}(\alpha'(t)\times\alpha''(t))\cdot\alpha''(t) + \frac{(\alpha'(t)\cdot\alpha''(t))^2}{|\alpha'(t)|^{10}}(\alpha'(t)\times\alpha''(t))\cdot\alpha'(t) \\
&= \frac{1}{|\alpha'(t)|^6}(\alpha'(t)\times\alpha'''(t))\cdot\alpha''(t).
\end{aligned}
$$

Fourth, from part b we get

$$\begin{aligned}|\beta''(s)|^2 &= (k(s))^2 = (k(s(t)))^2 = (k(t))^2 \\ &= \left(\frac{|\alpha'(t)\times\alpha''(t)|}{|\alpha'(t)|^3}\right)^2 \\ &= \frac{|\alpha'(t)\times\alpha''(t)|^2}{|\alpha'(t)|^6}.\end{aligned}$$

Therefore, our expression for torsion becomes

$$\begin{aligned}\tau(t) &= \frac{1}{|\beta''(s)|^2}(\beta'(s)\times\beta'''(s))\cdot\beta''(s) \\ &= \frac{1}{\frac{|\alpha'(t)\times\alpha''(t)|^2}{|\alpha'(t)|^6}}\frac{1}{|\alpha'(t)|^6}(\alpha'(t)\times\alpha'''(t))\cdot\alpha''(t) \\ &= \frac{(\alpha'(t)\times\alpha'''(t))\cdot\alpha''(t)}{|\alpha'(t)\times\alpha''(t)|^2},\end{aligned}$$

as desired. □

d. If $\alpha : I \to \mathbb{R}^2$ is a plane curve $\alpha(t) = (x(t), y(t))$, the signed curvature of α at t is

$$k(t) = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{\frac{3}{2}}}.$$

Proof. For ease of notation, we will write $(x, y), (x', y'), (x'', y'')$ to denote $(x(t), y(t)), (x'(t), y'(t)), (x''(t), y''(t))$, respectively. We have the tangent vector

$$\begin{aligned}t(s) &= \beta'(s) \\ &= \frac{\alpha'(t)}{|\alpha'(t)|} \\ &= \frac{(x', y')}{\sqrt{(x')^2 + (y')^2}},\end{aligned}$$

which implies we must have

$$n(s) = \frac{(-y', x')}{\sqrt{(x')^2 + (y')^2}}$$

in order to satisfy $|n(s)| = 1$ and

$$\begin{aligned}t(s)\cdot n(s) &= \frac{(x', y')}{\sqrt{(x')^2 + (y')^2}}\cdot\frac{(-y', x')}{\sqrt{(x')^2 + (y')^2}} \\ &= \frac{1}{(x')^2 + (y')^2}(x', y')\cdot(-y', x') \\ &= \frac{1}{(x')^2 + (y')^2}(-x'y' + y'x') \\ &= 0.\end{aligned}$$

Since $\alpha'(t) \cdot \alpha''(t) = (x', y') \cdot (x'', y'') = x'x'' + y'y''$, we have

$$\begin{aligned}
t'(s) &= \beta''(s) \\
&= \frac{\alpha''(t)}{|\alpha'(t)|^2} - \frac{(\alpha'(t) \cdot \alpha''(t))\alpha'(t)}{|\alpha'(t)|^4} \\
&= \frac{(x'', y'')}{(x')^2 + (y')^2} - \frac{(x'x'' + y'y'')(x', y')}{((x')^2 + (y')^2)^2} \\
&= \frac{(x')^2 + (y')^2}{(x')^2 + (y')^2} \frac{(x'', y'')}{(x')^2 + (y')^2} - \frac{(x'x'' + y'y'')(x', y')}{((x')^2 + (y')^2)^2} \\
&= \frac{((x')^2 + (y')^2)(x'', y'') - (x'x'' + y'y'')(x', y')}{((x')^2 + (y')^2)^2} \\
&= \frac{(((x')^2 + (y')^2)x'', ((x')^2 + (y')^2)y'') - ((x'x'' + y'y'')x', (x'x'' + y'y'')y')}{((x')^2 + (y')^2)^2} \\
&= \frac{((x')^2x'' + (y')^2x'', (x')^2y'' + (y')^2y'') - ((x')^2x'' + y'y''x', x'x''y' + (y')^2y'')}{((x')^2 + (y')^2)^2} \\
&= \frac{(((x')^2x'' + (y')^2x'') - ((x')^2x'' + y'y''x'), ((x')^2y'' + (y')^2y'') - (x'x''y' + (y')^2y''))}{((x')^2 + (y')^2)^2} \\
&= \frac{((y')^2x'' - y'y''x', (x')^2y'' - x'x''y')}{((x')^2 + (y')^2)^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
k(t) = k(t(s)) = k(s) &= t'(s) \cdot n(s) \\
&= \frac{((y')^2x'' - y'y''x', (x')^2y'' - x'x''y')}{((x')^2 + (y')^2)^2} \cdot \frac{(-y', x')}{\sqrt{(x')^2 + (y')^2}} \\
&= \frac{((y')^2x'' - y'y''x', (x')^2y'' - x'x''y') \cdot (-y', x')}{((x')^2 + (y')^2)^{\frac{5}{2}}} \\
&= \frac{((y')^2x'' - y'y''x')(-y') + ((x')^2y'' - x'x''y')x'}{((x')^2 + (y')^2)^{\frac{5}{2}}} \\
&= \frac{-(y')^2y'x'' + (y')^2y''x' + (x')^2y''x' - (x')^2x''y'}{((x')^2 + (y')^2)^{\frac{5}{2}}} \\
&= \frac{-(y')^2y'x'' - (x')^2x''y' + (x')^2y''x' + (y')^2y''x'}{((x')^2 + (y')^2)^{\frac{5}{2}}} \\
&= \frac{-y'x''((y')^2 + (x')^2) + y''x'((x')^2 + (y')^2)}{((x')^2 + (y')^2)^{\frac{5}{2}}} \\
&= \frac{(y''x' - y'x'')((x')^2 + (y')^2)}{((x')^2 + (y')^2)^{\frac{5}{2}}} \\
&= \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{\frac{3}{2}}},
\end{aligned}$$

as desired. □

1-5.13. Assume that $\tau(s) \neq 0$ and $k'(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$R^2 + (R')^2T^2 = \text{const.},$$

where $R = \frac{1}{k}$, $T = \frac{1}{\tau}$, and R' is the derivative of R relative to s.

Proof. Recall that the phrase "necessary and sufficient condition" is just another way of saying "if and only if". We can rewrite this exercise with an equivalent statement: Show that $\alpha(I)$ lies on a sphere if and only if $R^2 + (R')^2T^2 = \text{const.}$.

We will first prove the forward direction: If $\alpha(I)$ lies on a sphere, then $R^2 + (R')^2T^2 = \text{const.}$. To prove this forward direction, first let p be a fixed point which is the center of a sphere. Since $\alpha(I)$ lies on this sphere with center p, we have $|\alpha(s) - p| = 1$, or equivalently $|\alpha(s) - p|^2 = 1$, for all $s \in I$. Taking the derivative in s of both sides of $|\alpha(s) - p|^2 = 1$, we obtain

$$\frac{d}{ds}|\alpha(s) - p|^2 = \frac{d}{ds}(1),$$

or equivalently

$$\alpha'(s) \cdot (\alpha(s) - p) = 0.$$

Taking the derivative in s of both sides of $\alpha'(s) \cdot (\alpha(s) - p) = 0$, we get

$$\frac{d}{ds}\alpha'(s) \cdot (\alpha(s) - p) + \alpha'(s) \cdot \frac{d}{ds}(\alpha(s) - p) = 0,$$

or equivalently

$$\alpha''(s) \cdot (\alpha(s) - p) + \alpha'(s) \cdot \alpha'(s) = 0.$$

Consequently, we get

$$\begin{aligned} 0 &= \alpha''(s) \cdot (\alpha(s) - p) + \alpha'(s) \cdot \alpha'(s) \\ &= \alpha''(s) \cdot (\alpha(s) - p) + |\alpha'(s)|^2 \\ &= k(s)n(s) \cdot (\alpha(s) - p) + 1, \end{aligned}$$

from which we get

$$-\frac{1}{k(s)} = n(s) \cdot (\alpha(s) - p)$$

But we have $R(s) = \frac{1}{k(s)}$ from the hypotheses, so we really get

$$-R(s) = n(s) \cdot (\alpha(s) - p).$$

Now, taking the derivative in s of both sides of $k(s)n(s) \cdot (\alpha(s) - p) + 1 = 0$, we get

$$\frac{d}{ds}k(s)(n(s) \cdot (\alpha(s) - p)) + k(s)\frac{d}{ds}(n(s) \cdot (\alpha(s) - p)) = 0,$$

or equivalently

$$k'(s)(n(s) \cdot (\alpha(s) - p)) + k(s)(n'(s) \cdot (\alpha(s) - p) + n(s) \cdot \alpha'(s)) = 0.$$

Since $\alpha'(s)$ is perpendicular to $n(s)$ (i.e. $n(s) \cdot \alpha'(s) = 0$) and since we already established $n(s) \cdot (\alpha(s) - p) = -\frac{1}{k(s)}$ from moments earlier, our equation becomes

$$-\frac{k'(s)}{k(s)} + k(s)n'(s) \cdot (\alpha(s) - p) = 0.$$

Consequently, applying the Frenet formulas and realizing from the result of taking our first derivative that $t(s) \cdot (\alpha(s) - p) = \alpha'(s) \cdot (\alpha(s) - p) = 0$, we get

$$\begin{aligned} 0 &= -\frac{k'(s)}{k(s)} + k(s)n'(s) \cdot (\alpha(s) - p) \\ &= -\frac{k'(s)}{k(s)} + k(s)(-k(s)t(s) - \tau(s)b(s)) \cdot (\alpha(s) - p) \\ &= -\frac{k'(s)}{k(s)} - (k(s))^2 t(s) \cdot (\alpha(s) - p) - k(s)\tau(s)b(s) \cdot (\alpha(s) - p) \\ &= -\frac{k'(s)}{k(s)} - k(s)\tau(s)b(s) \cdot (\alpha(s) - p), \end{aligned}$$

from which we conclude that

$$-\frac{k'(s)}{(k(s))^2}\frac{1}{\tau(s)} = b(s) \cdot (\alpha(s) - p).$$

Since we were given $R(s) = \frac{1}{k(s)}$ (which implies its first derivative $R'(s) = -\frac{k'(s)}{(k(s))^2}$) and $T(s) = \frac{1}{\tau(s)}$, our equation becomes

$$R'(s)T(s) = b(s) \cdot (\alpha(s) - p).$$

After having estalished so far the equalities $-R(s) = n(s) \cdot (\alpha(s) - p)$ and $R'(s)T(s) = b(s) \cdot (\alpha(s) - p)$, we would now like to turn our attention to writing our expression of $\alpha(s) - p$. We recall from earlier the equation $t(s) \cdot (\alpha(s) - p) = 0$, as well as the fact that $t(s)$ is perpendicular to both $n(s)$ and $b(s)$ (i.e. $t(s) \cdot n(s) = 0$ and $t(s) \cdot b(s) = 0$). So we can actually write $\alpha(s) - p = R(s)n(s) + R'(s)T(s)b(s)$ because it satisfies

$$\begin{aligned} t(s) \cdot (\alpha(s) - p) &= t(s) \cdot (R(s)n(s) + R'(s)T(s)b(s)) \\ &= R(s)t(s) \cdot n(s) + R'(s)T(s)t(s) \cdot b(s) \\ &= 0. \end{aligned}$$

Therefore, from $\alpha(s) - p = R(s)n(s) + R'(s)T(s)b(s)$, we obtain its associated square of the magnitude

$$\begin{aligned} |\alpha(s) - p|^2 &= (R(s))^2 + (R'(s)T(s))^2 \\ &= (R(s))^2 + (R'(s))^2(T(s))^2 \end{aligned}$$

But we also have from the beginning that $|\alpha(s) - p|^2$ is constant and therefore we conclude that $(R(s))^2 + (R'(s))^2(T(s))^2 =$ const., which completes the proof of the forward direction.

Now we will prove the backward direction: If $(R(s))^2 + (R'(s))^2(T(s))^2 =$ const., then $\alpha(I)$ lies on a sphere. To this end, suppose we have $(R(s))^2 + (T(s)R'(s))^2 =$ const.. Taking the derivative in s of both sides of $(R(s))^2 + (T(s)R'(s))^2 =$ const., we get

$$\frac{d}{ds}((R(s))^2 + (T(s)R'(s))^2) = \frac{d}{ds}(\text{const.}),$$

or equivalently

$$2R(s)R'(s) + 2T(s)R'(s)(T(s)R'(s))' = 0.$$

Consequently, as we have $T(s) = \frac{1}{\tau(s)}$ from the hypotheses, we get

$$\begin{aligned} 0 &= 2R(s)R'(s) + 2T(s)R'(s)(T(s)R'(s))' \\ &= 2R(s)R'(s) + 2\frac{1}{\tau(s)}R'(s)(T(s)R'(s))' \\ &= \frac{2R'(s)}{\tau(s)}(R(s)\tau(s) + (T(s)R'(s))'), \end{aligned}$$

which is fine since we assumed $\tau(s) \neq 0$ and $k'(s) \neq 0$ in the hypotheses, the latter of which implies $R'(s) = -\frac{k'(s)}{(k(s))^2} \neq 0$. So we can divide both sides by $\frac{2R'(s)}{\tau(s)}$ of our above equation to conclude that

$$R(s)\tau(s) + (T(s)R'(s))' = 0.$$

Next, we define $\beta(s) = \alpha(s) + R(s)n(s) - R'(s)T(s)b(s)$. Then we obtain its derivative

$$\begin{aligned} \beta'(s) = \frac{d\beta}{ds} &= \frac{d}{ds}(\alpha(s) + R(s)n(s) - R'(s)T(s)b(s)) \\ &= \frac{d}{ds}\alpha(s) + \frac{d}{ds}(R(s)n(s)) - \frac{d}{ds}(R'(s)T(s)b(s)) \\ &= \alpha'(s) + (R'(s)n(s) + R(s)n'(s)) - ((R'(s)T(s))'b(s) + R'(s)T(s)b'(s)) \\ &= t(s) + (R'(s)n(s) + R(s)(-k(s)t(s) - \tau(s)b(s)) - ((R'(s)T(s))'b(s) + R'(s)T(s)(\tau(s)n(s)) \\ &= t(s) + R'(s)n(s) - R(s)k(s)t(s) - R(s)\tau(s)b(s) - (R'(s)T(s))'b(s) - R'(s)T(s)\tau(s)n(s) \\ &= t(s) + R'(s)n(s) - \frac{1}{k(s)}k(s)t(s) - R(s)\tau(s)b(s) - (R'(s)T(s))'b(s) - R'(s)\frac{1}{\tau(s)}\tau(s)n(s) \\ &= t(s) + R'(s)n(s) - t(s) - R(s)\tau(s)b(s) - (T(s)R'(s))'b(s) - R'(s)n(s) \\ &= -R(s)\tau(s)b(s) - (T(s)R'(s))'b(s) \\ &= -(R(s)\tau(s) + (T(s)R'(s))')b(s) \\ &= 0, \end{aligned}$$

from which we conclude that $\beta(s)$ is constant (i.e. $\beta(s) = p$, where p is constant in s). From $\beta(s) = p$, we conclude

$$\alpha(s) + R(s)n(s) - R'(s)T(s)b(s) = p,$$

or

$$\alpha(s) - p = -R(s)n(s) + R'(s)T(s)b(s).$$

So we obtain the square of the magnitude

$$\begin{aligned} |\alpha(s) - p|^2 &= (-R(s))^2 + (R'(s)T(s))^2 \\ &= (R(s))^2 + (T(s)R'(s))^2 \\ &= \text{const.}, \end{aligned}$$

which means $\alpha(I)$ lies on a sphere of center p and therefore completes the proof of the backward direction. □

1-5.14. Let $\alpha : (a, b) \to \mathbb{R}^2$ be a regular parametrized plane curve. Assume that there exists t_0, $a < t_0 < b$, such that the distance $|\alpha(t)|$ from the origin to the trace of α will be a maximum at t_0. Prove that the curvature k of α at t_0 satisfies $|k(t_0)| \geq \frac{1}{|\alpha(t_0)|}$.

Proof. Because $|\alpha(t)|$ is maximum (and therefore $|\alpha(t)|^2$ is maximum at $t = t_0$, we have $\frac{d}{dt}(|\alpha(t_0)|^2) = 0$, or equivalently

$$\alpha'(t) \cdot \alpha(t)|_{t=t_0} = 0.$$

Taking the derivative in t of both sides of $\alpha'(t) \cdot \alpha(t) = 0$ and then setting $t = t_0$, we get

$$\left(\frac{d}{dt}\alpha'(t) \cdot \alpha(t) + \alpha'(t) \cdot \frac{d}{dt}\alpha(t)\right)\Bigg|_{t=t_0} = 0,$$

or equivalently

$$\alpha''(t_0) \cdot \alpha(t_0) + \alpha'(t_0) \cdot \alpha'(t_0) = 0.$$

Consequently, we have

$$\begin{aligned} 0 &= \alpha''(t_0) \cdot \alpha(t_0) + \alpha'(t_0) \cdot \alpha'(t_0) \\ &= \alpha''(t_0) \cdot \alpha(t_0) + |\alpha'(t_0)|^2 \\ &= \alpha''(t_0) \cdot \alpha(t_0) + 1 \\ &= |\alpha''(t_0)||\alpha(t_0)| \cos\theta + 1 \\ &= k(t_0)|\alpha(t_0)| \cos\theta + 1, \end{aligned}$$

from which we get $k(t_0) = -\frac{1}{|a(t_0)|\cos\theta}$. Since $\cos\theta \geq -1$, which implies $\frac{1}{\cos\theta} \geq -1$, we get

$$\begin{aligned} k(t_0) &= -\frac{1}{|a(t_0)|\cos\theta} \\ &\geq \frac{1}{|a(t_0)|}, \end{aligned}$$

as desired. □