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Solutions to assigned homework problems from *Differential Geometry of Curves and Surfaces* by Manfredo Perdigão do Carmo

Assignment 3 – pages 47: 1,2,5

1-7.1. Is there a simple closed curve in the plane with length equal to 6 feet and bounding an area of 3 square feet?

Proof. Since C is a simply closed curve of some length l and bounding some area A, we must have the Isoperimetric Inequality $l^2 - 4\pi A \geq 0$ (c.f. do Carmo, page 33). However, if $l = 6$ and $A = 3$, then

$$\begin{aligned} l^2 - 4\pi A &= (6)^2 - 4\pi(3) \\ &= 36 - 12\pi \\ &= 12(3-\pi) \\ &< 0 \end{aligned}$$

since $3 < \pi$. So the Isoperimetric Inequality is not satisfied, which means there does not exist such a simple closed curve with $l = 6$, $A = 3$. □

1-7.2. Let $\overline{AB}$ be a segment of a straight line and let $l >$ length of AB. Show that the curve C joining A and B, with length l, and such that together with $\overline{AB}$ bounds the largest possible area, is an arc of a circle passing through A and B.

Proof. Let S^1 be a circle such that $\overline{AB}$ is a chord of S^1. (Recall that a chord of a circle is any line segment that connects two points on a circle.) Let α and β be two curves which together comprise a closed curve that passes through both endpoints of $\overline{AB}$. Then one of the two arcs has length greater than l; suppose without loss of generality that β is a fixed arc and α is the curve with its length being greater than l. Now we consider a piecewise C^1 curve, which is one-time differentiable except at finitely many points (such as at some sharp corners of the curve). Then Remark 2 (c.f. do Carmo, page 35) states that the Isoperimetric Inequality also holds for piecewise C^1 curves; therefore, according to the Isoperimetric Inequality, the largest area in the curve formed by both α and β occurs if and only if α and β comprise a circle passing through the endpoints of $\overline{AB}$. In particular, α must be an arc of a circle passing through the endpoints of $\overline{AB}$. Since we stated already that β is a fixed arc, it follows that α together with segment $\overline{AB}$ bounds the largest possible area. □

1-7.5. If a closed plane curve C is contained inside a disk of radius r, prove that there exists a point $p \in C$ such that the curvature k of C at p satisfies $|k| \geq \frac{1}{r}$.

Proof. This problem assumes that the disk of radius r is centered about the origin, and moreover that C is a closed plane curve contained inside the disk. In other words, if we assume t to be the arc length parametrization of $\alpha(t)$ (i.e. $|\alpha'(t)| = 1$), the assumptions imply that we have $|\alpha(t)| \leq r$ for all $t \in I$. Now, we obtain the first derivative

$$\begin{aligned} \frac{d}{dt}(|\alpha(t)|^2) &= \frac{d}{dt}(\alpha(t)\cdot\alpha(t)) \\ &= \alpha'(t)\cdot\alpha(t) + \alpha(t)\cdot\alpha'(t) \\ &= 2\alpha(t)\cdot\alpha'(t) \end{aligned}$$

and the second derivative

$$\begin{aligned} \frac{d^2}{dt^2}(|\alpha(t)|^2) &= \frac{d}{dt}\left(\frac{d}{dt}(\alpha(t)\cdot\alpha(t))\right) \\ &= \frac{d}{dt}(2\alpha'(t)\cdot\alpha'(t)) \\ &= 2(\alpha''(t)\cdot\alpha(t) + \alpha'(t)\cdot\alpha'(t)) \\ &= 2(\alpha''(t)\cdot\alpha(t) + |\alpha'(t)|^2) \\ &= 2(\alpha''(t)\cdot\alpha(t) + 1). \end{aligned}$$

Let $t_0 \in I$ be some critical value at which $|\alpha(t)|$ is maximum (or equivalently $|\alpha(t)|^2$ is maximum). Then $\frac{d}{dt}(|\alpha(t_0)|^2) = 0$ by the first derivative test and $\frac{d^2}{dt^2}(|\alpha(t_0)|^2) \leq 0$ by the second derivative test. Applying our expressions above, we conclude that $\alpha(t_0)\cdot\alpha'(t_0) = 0$ and $\alpha''(t_0)\cdot\alpha(t_0) + 1 \leq 0$. Also, we can take the derivative in t of both sides of $|\alpha'(t)|^2 = 1$, using the product rule while doing so (similar to what we did in Exercise 1-2.5), to obtain eventually $\alpha''(t)\cdot\alpha'(t) = 0$; in particular, we have $\alpha''(t_0)\cdot\alpha'(t_0) = 0$. From the last two sentences, we established that $\alpha(t_0)$ is perpendicular to $\alpha'(t_0)$ and that $\alpha'(t_0)$ is perpendicualr to $\alpha''(t_0)$. This implies that, in a plane, the angle θ between $\alpha(t_0)$ and $\alpha''(t_0)$ must be either 0 or π; as we have $\theta = 0$ or $\theta = \pi$, we get $\cos\theta = \pm 1$. Therefore, we have

$$\begin{aligned} \alpha''(t_0)\cdot\alpha(t_0) &= |\alpha''(t_0)||\alpha(t_0)|\cos\theta \\ &= \pm|\alpha''(t_0)||\alpha(t_0)| \\ &= \pm k(t_0)|\alpha(t_0)|. \end{aligned}$$

So $\alpha''(t_0) \cdot \alpha(t_0) + 1 \leq 0$ really becomes $\pm k(t_0)|\alpha(t_0)| + 1 \leq 0$, or $\pm k(t_0)|\alpha(t_0)| \leq -1$. Splitting up the $\pm$ sign, we have either $k(t_0)|\alpha(t_0)| \leq -1$ or $-k(t_0)|\alpha(t_0)| \leq -1$; equivalently, either $k(t_0) \leq -\frac{1}{|\alpha(t_0)|}$ or $k(t_0) \geq \frac{1}{|\alpha(t_0)|}$. The first inequality $k(t_0) \leq -\frac{1}{|\alpha(t_0)|}$ is nonsense because the curvature $k(t_0)$ is always nonnegative. So we must have the second inequality $k(t_0) \geq \frac{1}{|\alpha(t_0)|}$. Finally, we said $|\alpha(t)| \leq r$ at the beginning of this proof, we have in particular $|\alpha(t_0)| \leq r$, or equivalently $\frac{1}{|\alpha(t_0|} \geq \frac{1}{r}$. Combining our last two results together, we conclude that

$$\begin{aligned} k(t_0) &\geq \frac{1}{|\alpha(t_0)|} \\ &\geq \frac{1}{r}, \end{aligned}$$

as desired. □