MATH 138A discussion
University of California, Riverside

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Solutions to assigned homework problems from *Differential Geometry of Curves and Surfaces* by Manfredo Perdigão do Carmo

Assignment 4 – pages 88-92: 1,2,3,4,11,15

2-4.1. Show that the equation of the tangent plane at (x_0, y_0, z_0) of a regular surface given by $f(x, y, z) = 0$, where 0 is a regular valueo f f, is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Proof. Let $\alpha(t) = (x(t), y(t), z(y))$ be a curve on the surface passing through (x_0, y_0, z_0) at $t = 0$. We can take the derivative in t of both sides of $f(x, y, z) = 0$, using the multivariable chain rule in doing so, to get

$$f_x(x, y, z)x'(t) + f_y(x, y, z)y'(t) + f_z(x, y, z)z'(t) = 0._0, y_0,$$

Equivalently, we have

$$\begin{aligned}\nabla f(x, y, z) \cdot \alpha'(t) &= (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)) \cdot (x'(t), y'(t), z'(t))\\ &= f_x(x, y, z)x'(t) + f_y(x, y, z)y'(t) + f_z(x, y, z)z'(t)\\ &= 0.\end{aligned}$$

In particular, at $t = 0$, we have

$$\nabla f(x_0, y_0, z_0) \cdot \alpha'(0) = 0.$$

Since our choice of $\alpha(t)$ passing through p at $t = 0$ is arbitrary, $\alpha'(0)$ is also arbitrary. In other words, all vectors tangent to the surface at (x_0, y_0, z_0) must be perpendicular to $\nabla f(x_0, y_0, z_0)$. In particular, if (x, y, z) is another point in the plane tangent to the surface at (x_0, y_0, z_0), then the vector $(x - x_0, y - y_0, z - z_0)$ lies in that tangent plane, and so we have

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

or equivalently,

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0,$$

as desired. □

2-4.2. Determine the tangent planes of $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0)$ and show that they are all parallel to the z-axis.

Proof. Let $f(x, y, z) = x^2 + y^2 - z^2$. Then $f_x(x, y, z) = 2x$, $f_y(x, y, z) = 2y$, $f_z(x, y, z) = -2z$; in particular, at the point $(x_0, y_0, 0)$, we have $f_x(x_0, y_0, 0) = 2x_0$, $f_y(x_0, y_0, 0) = 2y_0$, $f_z(x_0, y_0, 0) = 0$. So the tangent equation at the point $(x_0, y_0, 0)$ is

$$f_x(x_0, y_0, 0)(x - x_0) + f_y(x_0, y_0, 0)(y - y_0) + f_z(x_0, y_0, 0)(z - 0) = 0,$$

or equivalently,

$$\begin{aligned}\nabla f(x_0, y_0, 0) \cdot (x - x_0, y - y_0, z - z_0) &= (f_x(x_0, y_0, 0), f_y(x_0, y_0, 0), f_z(x_0, y_0, 0)) \cdot (x - x_0, y - y_0, z - z_0)\\ &= f_x(x_0, y_0, 0)(x - x_0) + f_y(x_0, y_0, 0)(y - y_0) + f_z(x_0, y_0, 0)(z - z_0)\\ &= 2x_0(x - x_0) + 2y_0(y - y_0) - 0(z - 0)\\ &= 0,\end{aligned}$$

which signifies that $\nabla f(x_0, y_0, 0)$ is perpendicular to the tangent plane. Now, we consider the vectors $(0, 0, \pm 1)$ in the z-axis. Then we also have

$$\begin{aligned}\nabla f(x_0, y_0, 0) \cdot (0, 0, \pm 1) &= (f_x(x_0, y_0, 0), f_y(x_0, y_0, 0), f_z(x_0, y_0, 0)) \cdot (0, 0, \pm 1)\\ &= f_x(x_0, y_0, 0)(0) + f_y(x_0, y_0, 0)(0) + f_z(x_0, y_0, 0)(\pm 1)\\ &= (2x_0)(0) + (2y_0)(0) + (0)(\pm 1)\\ &= 0,\end{aligned}$$

which signifies that $\nabla f(x_0, y_0, 0)$ is also perpendicular to the z-axis. Therefore, the tangent plane and the z-axis are parallel. Since we argued this for the point $(x_0, y_0, 0)$, we can extend our argument to arbitrary points of the form $(x, y, 0)$, as desired. □

2-4.3. Show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z = f(x, y)$, at the point $p_0 = (x_0, y_0)$, is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Recall the definition of the differential df of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and show that the tangent plane is the graph of the differential df_p.

Proof. Define $g(x, y, z) := f(x, y) - z$. Then we are describing the set of all points (x, y, z) satisfying $g(x, y, z) = 0$ (that is, satisfying $z = f(x, y)$). We obtain the partial derivatives $g_x(x, y, z) = f_x(x, y)$, $g_y(x, y, z) = f_y(x, y)$, $g_z(x, y, z) = -1$. According to Exercise 2-4.1, the equation of the tangent plane (applied to $g(x, y, z)$) is

$$g_x(x_0, y_0, z_0)(x - x_0) + g_y(x_0, y_0, z_0)(y - y_0) + g_z(x_0, y_0, z_0)(z - z_0) = 0,$$

or equivalently

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - 1(z - f(x_0, y_0)) = 0.$$

Solving for z, we arrive at

$$z = f(x_0, y_0)(x - x_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

as desired. □

2-4.4. Show that the tangent planes of a surface given by $z = xf(\frac{y}{x})$, $x \neq 0$, where f is a differentiable function, all pass through the origin $(0, 0, 0)$.

Proof. Let $t = \frac{y}{x}$. Then $z(x, y) = xf(\frac{y}{x}) = xf(t)$, and so we obtain the partial derivatives

$$\begin{aligned}
z_x(x, y) = \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}(xf(t)) \\
&= f(t) + x\frac{\partial}{\partial x}f(t) \\
&= f(t) + xf'(t)\frac{\partial t}{\partial x} \\
&= f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right) \\
&= f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) \\
&= f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)
\end{aligned}$$

and

$$\begin{aligned}
z_y(x, y) = \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}(xf(t)) \\
&= x\frac{\partial}{\partial y}f(t) \\
&= xf'(t)\frac{\partial t}{\partial y} \\
&= xf'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right) \\
&= xf'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\frac{1}{x} \\
&= f'\left(\frac{y}{x}\right).
\end{aligned}$$

The tangent equation in Exercise 2-4.3 gives us

$$\begin{aligned}
z(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
&= z(x_0, y_0) + z_x(x_0, y_0)(x - x_0) + z_y(x_0, y_0)(y - y_0) \\
&= x_0 f\left(\frac{x_0}{y_0}\right) + \left(f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right)(x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0).
\end{aligned}$$

In particular, our equation of the tangent plane satisfies

$$\begin{aligned}
z(0, 0) &= x_0 f\left(\frac{x_0}{y_0}\right) + \left(f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right)(0 - x_0) + f'\left(\frac{y_0}{x_0}\right)(0 - y_0) \\
&= x_0 f\left(\frac{x_0}{y_0}\right) - f\left(\frac{y_0}{x_0}\right)x_0 + \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)x_0 - f'\left(\frac{y_0}{x_0}\right)y_0 \\
&= 0
\end{aligned}$$

which implies that the tangent plane goes through the point $(0, 0, 0)$ (that is, the origin). □

2-4.11. Show that the normals to a parametrized surface given by

$$\mathbf{x}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

$f(u) \neq 0$, $g' \neq 0$, all pass through the z-axis.

Proof. Given $\mathbf{x}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, we obtain the derivatives

$$\begin{aligned} \mathbf{x}_u(u,v) &= \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial}{\partial u}(f(u)\cos v, f(u)\sin v, g(u)) \\ &= \left(\frac{\partial}{\partial u}(f(u)\cos v), \frac{\partial}{\partial u}(f(u)\sin v), \frac{\partial}{\partial u}g(u)\right) \\ &= (f'(u)\cos v, f'(u)\sin v, g'(u)) \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_v(u,v) &= \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial}{\partial v}(f(u)\cos v, f(u)\sin v, g(u)) \\ &= \left(\frac{\partial}{\partial v}(f(u)\cos v), \frac{\partial}{\partial v}(f(u)\sin v), \frac{\partial}{\partial v}g(u)\right) \\ &= (-f(u)\sin v, f(u)\cos v, 0). \end{aligned}$$

So we obtain the cross product

$$\begin{aligned} \mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f'(u)\cos v & f'(u)\sin v & g'(u) \\ -f(u)\sin v & f(u)\cos v & 0 \end{vmatrix} \\ &= \begin{vmatrix} f'(u)\sin v & g'(u) \\ f(u)\cos v & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} f'(u)\cos v & g'(u) \\ -f(u)\sin v & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} f'(u)\cos v & f'(u)\cos v \\ -f(u)\sin v & f(u)\cos v \end{vmatrix}\mathbf{k} \\ &= ((f'(u)\sin v)(0) - (f(u)\cos v)(g'(u)))\mathbf{i} - ((f'(u)\cos v)(0) - (-f(u)\sin v)(g'(u)))\mathbf{j} \\ &\quad - ((f'(u)\cos v)(f(u)\cos v) - (-f(u)\sin v)(f'(u)\sin v))\mathbf{k} \\ &= (-f(u)g'(u)\cos v)\mathbf{i} + (-f(u)g'(u)\sin v)\mathbf{j} + (f(u)f'(u))\mathbf{k} \\ &= (-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f(u)f'(u)) \end{aligned}$$

and its associated magnitude

$$\begin{aligned} |\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)| &= \sqrt{(-f(u)g'(u)\cos v)^2 + (-f(u)g'(u)\sin v)^2 + (f(u)f'(u))^2} \\ &= \sqrt{(f(u)g'(u))^2(\cos^2 v + \sin^2 v) + (f(u)f'(u))^2} \\ &= \sqrt{(f(u)g'(u))^2 + (f(u)f'(u))^2} \\ &= \sqrt{(f(u))^2((g'(u))^2 + (f'(u))^2)} \\ &= |f(u)|\sqrt{(g'(u))^2 + (f'(u))^2}. \end{aligned}$$

According to the definition on page 87 of do Carmo, we obtain the unit normal vector

$$\begin{aligned} N &= \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)|} \\ &= \frac{(-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f(u)f'(u))}{|f(u)|\sqrt{(g'(u))^2 + (f'(u))^2}}. \end{aligned}$$

Since the problem assumed that $f(u) \neq 0$ and $g' \neq 0$, it follows that $|\mathbf{x}_u \times \mathbf{x}_v| > 0$, which means N cannot possibly be undefined. The line that contains N is given by

$$\begin{aligned} \alpha(t) &= \mathbf{x}(u,v) + tN \\ &= (f(u)\cos v, f(u)\sin v, g(u)) + t\frac{(-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f(u)f'(u))}{|f(u)|\sqrt{(g'(u))^2 + (f'(u))^2}} \end{aligned}$$

for all $t \in \mathbb{R}$. However, this expression of $\alpha(t)$ is quickly getting complicated. So we should consider a slight workaround. To this end, it is important to observe that, if the line described by $\alpha(t)$ contains N, the same line also contains the vector $|f(u)|\sqrt{(g'(u))^2 + (f'(u))^2}N = (-f(u)g'(u)\cos v, f(u)g'(u)\sin v, f(u)f'(u))$, which is a scalar multiple of N. So we can describe the same line as

$$\begin{aligned} \beta(t) &= \mathbf{x}(u,v) + t(|f(u)|\sqrt{(g'(u))^2 + (f'(u))^2}N) \\ &= (f(u)\cos v, f(u)\sin v, g(u)) + t(-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f(u)f'(u)) \\ &= (f(u)\cos v - tf(u)g'(u)\cos v, f(u)\sin v - tf(u)g'(u)\sin v, g(u) + tf(u)f'(u)) \\ &= (1 - tg'(u))f(u)\cos v, (1 - tg'(u))f(u)\sin v, g(u) + tf(u)f'(u)), \end{aligned}$$

which is a much easier expression to work with. Now, to prove that this line passes through the z-axis, we need to find some $t_0 \in \mathbb{R}$ such that $\beta(t_0) = (0, 0, k)$ for some $k \in \mathbb{R}$. To satisfy this condition, we must set the first and second coordinates of $\beta(t_0)$ equal to zero; that is, we must set

$$(1 - t_0 g'(u)) f(u) \cos v = 0$$
$$(1 - t_0 g'(u)) f(u) \sin v = 0,$$

from which we will solve for t_0. To this end, we can multiply both sides of the first equation by $\cos v$ and both sides of the second equation by $\sin v$ so that our system of equations becomes

$$(1 - t_0 g'(u)) f(u) \cos^2 v = 0$$
$$(1 - t_0 g'(u)) f(u) \sin^2 v = 0.$$

So we can add up the two equations to obtain $(1 - t_0 g'(u)) f(u) = 0$. Since $f(u) \neq 0$, we can divide both sides of our latest equation by $f(u)$ to obtain $1 - t_0 g'(u) = 0$, and so $t_0 = \frac{1}{g'(u)}$. Hence, we have $\beta(t_0) = (0, 0, g(u) + t_0 f(u) f'(u))$, which means that the line $\beta(t)$ crosses the z-axis at $t = t_0$. □

2-4.15. Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

Proof. Notice that this question is a three-dimensional analog of Exercise 1-5.4; the proof for Exercise 2-4.15 will be copied verbatim from Exercise 1-5.4 except for minor adjustments. Let $p = (x_0, y_0, z_0)$ be a fixed point and $n(s)$ a unit normal vector of the parametrized surface $\alpha(s) = (x(u(s), v(s)), y(u(s), v(s)), z(u(s), v(s)))$; that is, $\alpha'(s) \cdot n(s) = 0$. Since $n(s)$ passes through p, we have $\alpha(s) - p = \lambda n(s)$ for some scalar $\lambda \in \mathbb{R}$. So we have

$$\begin{aligned}
\frac{d}{ds}(|\alpha(s) - p|^2) &= \frac{d}{ds}((\alpha(s) - p) \cdot (\alpha(s) - p)) \\
&= \frac{d}{ds}(\alpha(s) - p) \cdot (\alpha(s) - p) + (\alpha(s) - p) \cdot \frac{d}{ds}(\alpha(s) - p) \\
&= 2\frac{d}{ds}(\alpha(s) - p) \cdot (\alpha(s) - p) \\
&= 2\alpha'(s) \cdot (\alpha(s) - p) \\
&= 2\alpha'(s) \cdot \lambda n(s) \\
&= 2\lambda \alpha'(s) \cdot n(s) \\
&= 0.
\end{aligned}$$

Hence, $|\alpha(s) - p|^2$ is constant, which means $|\alpha(s) - p|$ is constant. In other words, for all $s \in I$ the distance between $\alpha(s)$ and p is the same, which implies that the surface parametrized by $\alpha(s)$ is contained in a sphere of center p. □