MATH 138A discussion
University of California, Riverside

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Winter 2019

Solutions to assigned homework problems from *Differential Geometry of Curves and Surfaces* by Manfredo Perdigão do Carmo

Assignment 5 – pages 99-101: 1ab,5,11

2-5.1. Compute the first fundamental forms of the following parametrized surfaces where they are regular:

a. $\mathbf{x}(u,v) = (a\sin u\cos v, b\sin u\sin v, c\cos u)$

Proof. The partial derivatives are

$$\begin{aligned}\mathbf{x}_u(u,v) &= \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial}{\partial u}(a\sin u\cos v, b\sin u\sin v, c\cos u)\\ &= \left(\frac{\partial}{\partial u}(a\sin u\cos v), \frac{\partial}{\partial u}(b\sin u\sin v), \frac{\partial}{\partial u}(c\cos u)\right)\\ &= (a\cos u\cos v, b\cos u\sin v, -c\sin u)\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_v(u,v) &= \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial}{\partial v}(a\sin u\cos v, b\sin u\sin v, c\cos u)\\ &= \left(\frac{\partial}{\partial v}(a\sin u\cos v), \frac{\partial}{\partial v}(b\sin u\sin v), \frac{\partial}{\partial v}(c\cos u)\right)\\ &= (-a\sin u\sin v, b\sin u\cos v, 0).\end{aligned}$$

So, if $(u,v) \subset \mathbb{R}^2$ is a regular point, then

$$\begin{aligned}E(u,v) &= \mathbf{x}_u(u,v)\cdot\mathbf{x}_u(u,v)\\ &= (a\cos u\cos v, b\cos u\sin v, -c\sin u)\cdot(-a\sin u\sin v, b\sin u\cos v, 0)\\ &= (a\cos u\cos v)(a\cos u\cos v) + (b\cos u\sin v)(b\cos u\sin v) + (-c\sin u)(-c\sin u)\\ &= a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u\\ &= (a^2\cos^2 v + b^2\sin^2 v)\cos^2 u + c^2\sin^2 u\end{aligned}$$

as well as

$$\begin{aligned}F(u,v) &= \mathbf{x}_u(u,v)\cdot\mathbf{x}_v(u,v)\\ &= (a\cos u\cos v, b\cos u\sin v, -c\sin u)\cdot(-a\sin u\sin v, b\sin u\cos v, 0)\\ &= (a\cos u\cos v)(-a\sin u\cos v) + (b\cos u\sin v)(b\sin u\cos v) + (-c\sin u)(0)\\ &= -a^2\cos u\cos v\sin u\sin v + b^2\cos u\sin v\sin u\cos v\\ &= (b^2 - a^2)\cos u\cos v\sin u\sin v\end{aligned}$$

and

$$\begin{aligned}G(u,v) &= \mathbf{x}_v(u,v)\cdot\mathbf{x}_v(u,v)\\ &= (-a\sin u\sin v, b\sin u\cos v, 0)\cdot(-a\sin u\sin v, b\sin u\cos v, 0)\\ &= (-a\sin u\sin v)(-a\sin u\sin v) + (b\sin u\cos v)(b\sin u\cos v) + (0)(0)\\ &= a^2\sin^2 u\sin^2 v + b^2\sin^2 u\cos^2 v\\ &= (a^2\sin^2 v + b^2\cos^2 v)\sin^2 u.\end{aligned}$$

Therefore, if we parametrize $u = u(t)$, $v = v(t)$ in order to consider the parametrized curve $\alpha(t) = \mathbf{x}(u(t), v(t))$, then, following page 93 of do Carmo, we have at $t = 0$ that

$$\begin{aligned}I_p(\alpha'(0)) &= E(u,v)(u')^2 + 2F(u,v)u'v' + G(u,v)(v')^2\\ &= ((a^2\cos^2 v + b^2\sin^2 v)\cos^2 u + c^2\sin^2 u)(u')^2\\ &\quad + 2((b^2 - a^2)\cos u\cos v\sin u\sin v)u'v' + ((a^2\sin^2 v + b^2\cos^2 v)\sin^2 u)(v')^2\end{aligned}$$

is the first fundamental form. ◻

b. $\mathbf{x}(u,v) = (au\cos v, bu\sin v, u^2)$

Proof. The partial derivatives are

$$\begin{aligned}\mathbf{x}_u(u,v) &= \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial}{\partial u}(au\cos v, bu\sin v, u^2)\\ &= \left(\frac{\partial}{\partial u}(au\cos v), \frac{\partial}{\partial u}(bu\sin v), \frac{\partial}{\partial u}(u^2)\right)\\ &= (a\cos v, b\sin v, 2u)\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_v(u,v) &= \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial}{\partial v}(au\cos v, bu\sin v, u^2)\\ &= \left(\frac{\partial}{\partial v}(au\cos v), \frac{\partial}{\partial v}(bu\sin v), \frac{\partial}{\partial v}(u^2)\right)\\ &= (-au\sin v, bu\cos v, 0).\end{aligned}$$

So, if $(u, v) \subset \mathbb{R}^2$ is a regular point, then

$$\begin{aligned}E(u,v) &= \mathbf{x}_u(u,v)\cdot\mathbf{x}_u(u,v)\\ &= (a\cos v, b\sin v, 2u)\cdot(a\cos v, b\sin v, 2u)\\ &= (a\cos v)(a\cos v) + (b\sin v)(b\sin v) + (2u)(2u)\\ &= a^2\cos^2 v + b^2\sin^2 v + 4u^2\end{aligned}$$

as well as

$$\begin{aligned}F(u,v) &= \mathbf{x}_u(u,v)\cdot\mathbf{x}_v(u,v)\\ &= (a\cos v, b\sin v, 2u)\cdot(-au\sin v, bu\cos v, 0)\\ &= (a\cos v)(-au\sin v) + (b\sin v)(bu\cos v) + (2u)(0)\\ &= -a^2u\cos v\sin v + b^2u\sin v\cos v\\ &= (b^2-a^2)u\sin v\cos v\end{aligned}$$

and

$$\begin{aligned}G(u,v) &= \mathbf{x}_v(u,v)\cdot\mathbf{x}_v(u,v)\\ &= (-au\sin v, bu\cos v, 0)\cdot(-au\sin v, bu\cos v, 0)\\ &= (-au\sin v)(-au\sin v) + (bu\cos v)(bu\cos v) + (0)(0)\\ &= a^2u^2\sin^2 v + b^2u^2\cos^2 v.\end{aligned}$$

Therefore, if we parametrize $u = u(t)$, $v = v(t)$ in order to consider the parametrized curve $\alpha(t) = \mathbf{x}(u(t), v(t))$, then, following page 93 of do Carmo, we have at $t = 0$ that

$$\begin{aligned}I_p(\alpha'(0)) &= E(u,v)(u')^2 + 2F(u,v)u'v' + G(u,v)(v')^2\\ &= (a^2\cos^2 v + b^2\sin^2 v + 4u^2)(u')^2\\ &\quad + 2((b^2-a^2)u\sin v\cos v)u'v' + (a^2u^2\sin^2 v + b^2u^2\cos^2 v)(v')^2\end{aligned}$$

is the first fundamental form. □

2-5.5. Show that the area A of a bounded region R of the surface $z = f(x, y)$ is

$$A = \iint_Q \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy,$$

where Q is the normal projection of R onto the xy-plane.

Proof. In general, for any surface parameterized by $\mathbf{x}(u, v)$, we obtain the partial derivatives

$$\begin{aligned}\mathbf{x}_u(u,v) &= \frac{\partial}{\partial u}\mathbf{x}(u,v)\\ &= \frac{\partial}{\partial u}(x(u,v), y(u,v), z(u,v))\\ &= \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_v(u,v) &= \frac{\partial}{\partial v}\mathbf{x}(u,v)\\ &= \frac{\partial}{\partial v}(x(u,v), y(u,v), z(u,v))\\ &= \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).\end{aligned}$$

So we obtain the cross product

$$\begin{aligned}\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v) &= \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)\times\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}\end{vmatrix}\\ &= \begin{vmatrix}\frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}\\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}\end{vmatrix}\mathbf{i} - \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial z}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial z}{\partial v}\end{vmatrix}\mathbf{j} + \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}\mathbf{k}\\ &= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)\mathbf{i} - \left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\right)\mathbf{j} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\mathbf{k}\\ &= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)\mathbf{i} + \left(\frac{\partial x}{\partial v}\frac{\partial z}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\mathbf{j} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\mathbf{k}\\ &= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}, \frac{\partial x}{\partial v}\frac{\partial z}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\end{aligned}$$

and its associated magnitude

$$|\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)| = \sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\frac{\partial z}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2}.$$

Specifically, for the surface described by $z = f(x,y)$, we can consider the parametrization $x = u$, $y = v$, $z = f(u,v)$. With this parametrization, we can express the magnitude as

$$\begin{aligned}|\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)| &= \sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\frac{\partial z}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2}\\ &= \sqrt{\left(\frac{\partial v}{\partial u}\frac{\partial f}{\partial v} - \frac{\partial v}{\partial v}\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial u}{\partial v}\frac{\partial f}{\partial u} - \frac{\partial u}{\partial u}\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial u}{\partial u}\frac{\partial v}{\partial v} - \frac{\partial u}{\partial v}\frac{\partial v}{\partial u}\right)^2}\\ &= \sqrt{((0)(f_v) - (1)(f_u))^2 + ((0)(f_u) - (1)(f_v))^2 + ((1)(1) - (0)(0))^2}\\ &= \sqrt{f_u^2 + f_v^2 + 1}\\ &= \sqrt{f_x^2 + f_y^2 + 1}\\ &= \sqrt{1 + f_x^2 + f_y^2}.\end{aligned}$$

Therefore, according to Definition 2 of Section 2.4 (c.f. do Carmo, page 98), we have

$$\begin{aligned}A &= \iint_Q |\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)|\,du\,dv\\ &= \iint_Q \sqrt{1 + f_x^2 + f_y^2}\,du\,dv,\end{aligned}$$

as desired. □

2-5.11. Let S be a surface of revolution and C its generating curve (c.f. Example 4 of Section 2-3; do Carmo, page 76). Let S be the arc length of C and denote by $\rho = \rho(s)$ the distance to the rotation axis of the point of C corresponding to s.

a. Show that the area of S is

$$2\pi\int_0^l \rho(s)\,ds,$$

where l is the length of C.

Proof. Define the set $U = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2\pi, a < v < b\}$. From Example 4 of Section 2-3 (c.f. do Carmo, page 76), S is parametrized by the map $\mathbf{x} : \mathbb{R}^2 \to S$ defined by

$$\mathbf{x}(u, v) := (f(v) \cos u, f(v) \sin u, g(v)),$$

where $g(v)$ is an arbitrary function and $f(v) > 0$ (which means $|f(v)| = f(v)$). We obtain the partial derivatives

$$\begin{aligned}
\mathbf{x}_u(u, v) &= \frac{\partial}{\partial u}\mathbf{x}(u, v) = \frac{\partial}{\partial u}(f(v) \cos u, f(v) \sin u, g(v)) \\
&= \left(\frac{\partial}{\partial u}(f(v) \cos u), \frac{\partial}{\partial u}(f(v) \sin u), \frac{\partial}{\partial u}g(v)\right) \\
&= (-f(v) \sin u, f(v) \cos u, 0)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{x}_v(u, v) &= \frac{\partial}{\partial v}\mathbf{x}(u, v) = \frac{\partial}{\partial v}(f(v) \cos u, f(v) \sin u, g(v)) \\
&= \left(\frac{\partial}{\partial v}(f(v) \cos u), \frac{\partial}{\partial v}(f(v) \sin u), \frac{\partial}{\partial v}g(v)\right) \\
&= (f'(v) \cos u, f'(v) \sin u, g'(v))
\end{aligned}$$

So we obtain the cross product

$$\begin{aligned}
\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) &= (-f(v) \sin u, f(v) \cos u, 0) \times (f'(v) \cos u, f'(v) \sin u, g'(v)) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -f(v) \sin u & f(v) \cos u & 0 \\ f'(v) \cos u & f'(v) \sin u & g'(v) \end{vmatrix} \\
&= \begin{vmatrix} f(v) \cos u & 0 \\ f'(v) \sin u & g'(v) \end{vmatrix}\mathbf{i} - \begin{vmatrix} -f(v) \sin u & 0 \\ f'(v) \cos u & g'(v) \end{vmatrix}\mathbf{j} + \begin{vmatrix} -f(v) \sin u & f(v) \cos u \\ f'(v) \cos u & f'(v) \sin u \end{vmatrix}\mathbf{k} \\
&= ((f(v) \cos u)(g'(v)) - (f'(v) \sin u)(0))\mathbf{i} - ((-f(v) \sin u)(g'(v)) - (f'(v) \cos u)(0))\mathbf{j} \\
&\quad + ((-f(v) \cos u)(f'(v) \sin u) - (f'(v) \cos u)(f(v) \cos u)\mathbf{k} \\
&= (f(v)g'(v) \cos u)\mathbf{i} + (f(v)g'(v) \sin u)\mathbf{j} + (-f(v)f'(v))\mathbf{k} \\
&= (f(v)g'(v) \cos u, f(v)g'(v) \sin u, -f(v)f'(v))
\end{aligned}$$

and its associated magnitude

$$\begin{aligned}
|\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)| &= \sqrt{(f(v)g'(v) \cos u)^2 + (f(v)g'(v) \sin u)^2 + (-f(v)f'(v))^2} \\
&= \sqrt{(f(v))^2((g'(v))^2(\cos^2 u + \sin^2 u) + (f'(v))^2)} \\
&= |f(v)| \sqrt{(g'(v))^2 + (f'(v))^2} \\
&= f(v) \sqrt{(f'(v))^2 + (g'(v))^2} \\
&= f(v)|\alpha'(v)|,
\end{aligned}$$

provided we let $\alpha(v) := (f(v), g(v))$, which means $|\alpha'(v)| = \sqrt{(f'(v))^2 + (g'(v))^2}$. Finally, if we change variables via the arc length substitution $s = s(v) := \int_a^v |\alpha'(t)|\, dt$ (c.f. do Carmo, page 22), then $ds = \frac{ds}{dv}\, dv = \frac{d}{dv}(\int_a^v |\alpha'(t)|\, dt)\, dv = |\alpha'(v)|\, dv$. Also, substituting $v = a$ and $v = b$, we get $s(a) = \int_a^a |\alpha'(t)|\, dt = 0$ and $s(b) = \int_a^b |\alpha'(t)|\, dt = l$. So we have

$$\begin{aligned}
\iint_U |\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)|\, du\, dv &= \int_a^b \int_0^{2\pi} f(v)|\alpha'(v)|\, du\, dv \\
&= \left(\int_0^{2\pi} du\right)\left(\int_a^b f(v)|\alpha'(v)|\, dv\right) \\
&= 2\pi \int_a^b f(v)|\alpha'(v)|\, dv \\
&= 2\pi \int_{s(a)}^{s(b)} f(v)\, ds \\
&= 2\pi \int_0^l f(v)\, ds.
\end{aligned}$$

Finally, since we already introduced $s = s(v)$, we now let $v = v(s)$ be the inverse function of s (as was similarly done in the statement of Exercise 1-5.12 (c.f. do Carmo, page 25)). We recall that the statement of the exercise denotes $\rho(s)$ to be the distance to the rotation axis of the point of C with respect to the parameter s. But Example 4 from Section 2-3 describes $f(v)$ to be exactly the same thing, except with the different parameter v. So we can reparameterize f to get $f(v) = f(v(s)) = \rho(s)$. This means we can really write

$$\iint_U |\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)| \, du \, dv = 2\pi \int_0^l \rho(s) \, ds,$$

as desired. □

b. Apply part a to compute the area of a torus of revolution.

Proof. Define the set $U := \{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2\pi, 0 < v < 2\pi\}$. Example 5 of Section 2-5 (c.f. do Carmo, pages 98-99) defines the parametrization $\mathbf{x} : U \to \mathbb{R}^3$ for the torus to be

$$\mathbf{x}(u, v) := ((a + r\cos v)\cos u, (a + r\cos v)\sin u, r\sin v)$$

for some $r > 0$. In other words, when comparing this map with the more general map

$$\mathbf{x}(u, v) := (f(v)\cos u, f(v)\sin u, g(v))$$

from part a, we have the functions

$$\begin{aligned} f(v) &= a + r\cos v \\ g(v) &= r\sin v, \end{aligned}$$

and thus their associated derivatives

$$\begin{aligned} f'(v) &= \frac{df}{dv} = \frac{d}{dv}(a + r\cos v) \\ &= -r\sin v \end{aligned}$$

and

$$\begin{aligned} g'(v) &= \frac{dg}{dv} = \frac{d}{dv}(r\sin v) \\ &= r\cos v. \end{aligned}$$

So, according to part a, we get the magnitude

$$\begin{aligned} |\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)| &= f(v)\sqrt{(f'(v))^2 + (g'(v))^2} \\ &= (a + r\cos v)\sqrt{(-r\sin v)^2 + (r\cos v)^2} \\ &= (a + r\cos v)\sqrt{r^2(\sin^2 v + \cos^2 v)} \\ &= (a + r\cos v)r. \end{aligned}$$

Therefore,

$$\begin{aligned} \iint_U |\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)| \, du \, dv &= \int_0^{2\pi}\int_0^{2\pi} (a + r\cos v)r \, du \, dv \\ &= r\left(\int_0^{2\pi} du\right)\left(\int_0^{2\pi} a + r\cos v \, dv\right) \\ &= r(2\pi)(2\pi a) \\ &= 4\pi^2 ra \end{aligned}$$

is the surface area of the torus. □