MATH 138A discussion
University of California, Riverside

Ryan Ta
Winter 2019

Solutions to assigned homework problems from *Differential Geometry of Curves and Surfaces* by Manfredo Perdigão do Carmo

Assignment 6 – pages 151-153: 2,3,4,5,6,8ab

3-2.2. Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

Proof. Let S be a surface and P be a plane that is tangent to it on the curve $\alpha(t)$ for all $t \in I$, where I is an interval that contains 0 in this problem. To say that a surface is tangent to a plane along a curve is really saying that S and P intersect each other only at points on $\alpha(t)$. Let $p = \alpha(0)$ be some point on the curve. We observe that all the normal vectors $N(\alpha(t))$ must be parallel; in other words, if $t_1, t_2 \in I$ are arbitrary values, then $N(\alpha(t_1)), N(\alpha(t_2))$ point in the same direction. This implies that we have $N'(t) = 0$. But we have also $dN_{\alpha(t)}(\alpha'(t)) = N'(t)$ (c.f. do Carmo, page 145). So altogether we have

$$\begin{aligned} dN_{\alpha(t)}(\alpha'(t)) &= N'(t) \\ &= 0. \end{aligned}$$

As $\alpha(t)$ is a parametrized curve, we must have $\alpha'(t) \neq 0$; otherwise, if $\alpha'(t) = 0$, then $\alpha(t)$ would be constant for all $t \in I$ and therefore not a parameterized curve. So we must have $\alpha'(t) = 0$ for all $t \in I$; in particular, we have $\alpha'(0) \neq 0$. Also at $t = 0$, since again $\alpha(0) = p$ and we established already $dN_{\alpha(t)}(\alpha'(t)) = 0$, we have in particular

$$dN_p(\alpha'(0)) = 0.$$

It follows then (from linear algebra) that the kernel of dN_p is nontrivial, which in turn implies (also from linear algebra) that dN_p is not injective as a map; in the language of matrices, dN_p is not an invertible matrix, which means we must have $\det dN_p = 0$. Therefore, if we also have $dN_p = 0$, then p is a planar point. Otherwise, if we have instead $dN_p \neq 0$, then p is a parabolic point. □

3-2.3. Let $C \subset S$ be a regular curve on a surface S with Gaussian curvature $K > 0$. Show that the curvature k of C at p satisfies

$$k \geq \min\{|k_1|, |k_2|\},$$

where k_1 and k_2 are the principal curvatures of S at p.

Proof. The normal curvature k_n is given by Euler's formula (c.f. do Carmo, page 145)

$$k_n(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

Also, according to Definition 3 on page 141 of do Carmo, the definition of the normal curvature is $k_n = k\cos\theta$. Finally, we need to recall that, since curvature is positive (i.e. $k > 0$), we have $|k| = k$. We also need to use the known fact of $|\cos\theta| \leq 1$. Since $K > 0$ by hypothesis and $K = k_1 k_2$, it follows that the principal curvatures k_1, k_2 must be either both positive or both negative; if one were negative and the other were positive, then $K = k_1 k_2 < 0$, which would contradict our hypothesis. So k_1, k_2 having the same sign means in particular that we will never have to deal with, e.g. $\pm k_1 \cos^2\theta \mp k_2 \sin^2\theta$ (notice the upside-down "$\mp$"), which would not allow the desired inequality $k \geq \min\{|k_1|, |k_2|\}$ to follow from it. Therefore, we have

$$\begin{aligned} k &= |k| \\ &\geq |k\cos\theta| \\ &= |k_n| \\ &= \pm k_n \\ &= \pm(k_1\cos^2\theta + k_2\sin^2\theta) \\ &= \pm k_1\cos^2\theta \pm k_2\sin^2\theta \\ &= |k_1|\cos^2\theta + |k_2|\sin^2\theta \\ &\geq \min\{|k_1|, |k_2|\}\cos^2\theta + \min\{|k_1|, |k_2|\}\sin^2\theta \\ &= \min\{|k_1|, |k_2|\}(\cos^2\theta + \sin^2\theta) \\ &= \min\{|k_1|, |k_2|\}, \end{aligned}$$

as desired. □

3-2.4. Assume that a surface S has the property that $|k_1| \leq 1$, $|k_2| \leq 1$ everywhere. Is it true that the curvature k of a curve on S also satisfies $|k| \leq 1$?

Proof. We have the Gaussian curvature (c.f. do Carmo, page 146)

$$K = k_1 k_2.$$

We will prove that the claim in the problem statement is not true. Let S be a plane. Then on S we have $k_1 = k_2 = 0$, which satisfies $|k_1| = 0 \leq 1$ and $|k_2| = 0 \leq 1$. If we consider a circle with radius $r < 1$, then this circle is a (closed) curve on S that has curvature $k = \frac{1}{r} > 1$, which does not satisfy $|k| \leq 1$. □

3-2.5. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi} \int_0^\pi k_n(\theta)\, d\theta,$$

where $k_n(\theta)$ is the normal curvature at p along a diretion making an angle θ with fixed direction.

Proof. The normal curvature k_n is given by Euler's formula (c.f. do Carmo, page 145)

$$k_n(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

We also recall that the mean curvature is given by (c.f. do Carmo, page 146)

$$H = \frac{k_1 + k_2}{2}.$$

So we have

$$\begin{aligned}
\int_0^\pi k_n(\theta)\, d\theta &= \int_0^\pi k_1 \cos^2\theta + k_2 \sin^2\theta\, d\theta \\
&= \int_0^\pi k_1 \frac{1 + \cos(2\theta)}{2} + k_2 \frac{1 - \cos(2\theta)}{2}\, d\theta \\
&= \frac{k_1}{2} \int_0^\pi 1 + \cos(2\theta)\, d\theta + \frac{k_2}{2} \int_0^\pi 1 - \cos(2\theta)\, d\theta \\
&= \frac{k_1}{2} \left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^\pi + \frac{k_2}{2} \left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^\pi \\
&= \frac{k_1}{2} \left(\left(\pi + \frac{1}{2}\sin(2\pi)\right) - \left(0 + \frac{1}{2}\sin(2(0))\right)\right) + \frac{k_2}{2} \left(\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \left(0 - \frac{1}{2}\sin(2(0))\right)\right) \\
&= \frac{k_1}{2}\pi + \frac{k_1}{2}\pi \\
&= \pi \frac{k_1 + k_2}{2} \\
&= \pi H,
\end{aligned}$$

which implies algebraically

$$H = \frac{1}{\pi} \int_0^\pi k_n(\theta)\, d\theta$$

as desired. □

3-2.6. Show that the sum of the normal curvatures for any pair of orthogonal directions at a point $p \in S$ is constant.

Proof. Once again, according to Definition 4 (c.f. do Carmo, page 144), then the normal curvature k_n is given by Euler's formula (c.f. do Carmo, page 145)

$$k_n(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

Since θ is an angle that corresponds to some direction on S at p, it follows for instance that $\theta + \frac{\pi}{2}$ is an angle that corresponds to a direction on S at p that is perpendicular to the original direction determined by θ. If v is a directional vector that is perpendicular to n, then The normal curvature for this perpendicular direction is

$$\begin{aligned}
k_v(\theta) &= k_n\left(\theta + \frac{\pi}{2}\right) \\
&= k_1 \cos^2\left(\theta + \frac{\pi}{2}\right) + k_2 \sin^2\left(\theta + \frac{\pi}{2}\right) \\
&= k_1 \left(\cos\left(\theta + \frac{\pi}{2}\right)\right)^2 + k_2 \left(\sin\left(\theta + \frac{\pi}{2}\right)\right)^2 \\
&= k_1 \left(\cos(\theta)\cos\left(\frac{\pi}{2}\right) - \sin(\theta)\sin\left(\frac{\pi}{2}\right)\right)^2 + k_2 \left(\sin(\theta)\cos\left(\frac{\pi}{2}\right) + \cos(\theta)\sin\left(\frac{\pi}{2}\right)\right)^2 \\
&= k_1((\cos\theta)(0) - (\sin\theta)(1))^2 + k_2((\sin\theta)(0) + (\cos\theta)(1))^2 \\
&= k_1 \sin^2\theta + k_2 \cos^2\theta.
\end{aligned}$$

Therefore, the sum of the normal curvatures for any pair of orthogonal directions is given by

$$\begin{aligned} k_n(\theta) + k_v(\theta) &= (k_1 \cos^2\theta + k_2 \sin^2\theta) + (k_1 \sin^2\theta + k_2\cos^2\theta) \\ &= k_1(\cos^2\theta + \sin^2\theta) + k_2(\sin^2\theta + \cos^2\theta) \\ &= k_1 + k_2, \end{aligned}$$

which does not depend on θ and is therefore constant. □

3-2.8. Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces:

a. Paraboloid of revolution $z = x^2 + y^2$.

Proof. Consider the parametrization $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\mathbf{x}(u, v) = (u, v, u^2 + v^2)$. Then the partial derivatives are

$$\begin{aligned} \mathbf{x}_u(u, v) &= \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial}{\partial u}(u, v, u^2 + v^2) \\ &= \left(\frac{\partial}{\partial u}(u), \frac{\partial}{\partial u}(v), \frac{\partial}{\partial u}(u^2 + v^2)\right) \\ &= (1, 0, 2u) \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_v(u, v) &= \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial}{\partial v}(u, v, u^2 + v^2) \\ &= \left(\frac{\partial}{\partial v}(u), \frac{\partial}{\partial v}(v), \frac{\partial}{\partial v}(u^2 + v^2)\right) \\ &= (0, 1, 2v). \end{aligned}$$

Then we obtain the cross product

$$\begin{aligned} \mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} \\ &= \begin{vmatrix} 0 & 2u \\ 1 & 2v \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 2u \\ 0 & 2v \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k} \\ &= ((0)(2v) - (1)(2u))\mathbf{i} - ((1)(2v) - (0)(2u))\mathbf{j} + ((1)(0) - (0)(1))\mathbf{k} \\ &= (-2u)\mathbf{i} + (-2v)\mathbf{j} + (1)\mathbf{k} \\ &= (-2u, -2v, 1) \end{aligned}$$

and its associated magnitude

$$\begin{aligned} |\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)| &= \sqrt{(-2u)^2 + (-2v)^2 + (1)^2} \\ &= \sqrt{4u^2 + 4v^2 + 1}. \end{aligned}$$

Thus, according to Definition 1 of Section 3-2 (c.f. page 136 of do Carmo), the normal vector $N : S \to S^2$, where $S \subset \mathbb{R}^3$ is a surface and $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is the unit sphere centered at the origin, is given by

$$\begin{aligned} N(u, v) &= \frac{\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)}{|\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)|} \\ &= \frac{(-2u, -2v, 1)}{\sqrt{4u^2 + 4v^2 + 1}} \\ &= \left(-\frac{2u}{\sqrt{4u^2 + 4v^2 + 1}}, -\frac{2v}{\sqrt{4u^2 + 4v^2 + 1}}, \frac{1}{\sqrt{4u^2 + 4v^2 + 1}}\right). \end{aligned}$$

At this point, we make two critical observations here. Our first observation is that the z-coordinate of our normal vector N—call this N_z—is positive; indeed, this is because we have $N_z := \frac{1}{\sqrt{4u^2+4v^2+1}} > 0$ for all $(u, v) \in \mathbb{R}^2$. This implies that the image of N must be contained in the upper hemisphere $H^+ := \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 + z^2 = 1, z > 0\} \subset S^2$. Our second observation is that the first two coordinates $N_x := -\frac{2u}{\sqrt{4u^2+4v^2+1}}$ and $N_y := -\frac{2v}{\sqrt{4u^2+4v^2+1}}$ of N are arbitrary real numbers depending on $(u, v) \in \mathbb{R}^2$; the significance of this fact is that we can conclude that the image of N is not part of H^+ (that is, properly contained in some strict subset of H^+), but rather the image of N is actually equal to H^+ itself. □

b. Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$.

Note: For part b, I presented two solutions here. The reader is recommended to only follow Solution 1 because Solution 2 is long, difficult, and redundant.

Solution 1: Gradient of a function

Proof. Define $f(x, y, z) := x^2 + y^2 - z^2 - 1$. Then the partial derivatives are $f_x(x, y, z) = 2x$, $f_y(x, y, z) = 2y$, $f_z(x, y, z) = -2z$, and so we obtain the gradient

$$\begin{aligned}\nabla f(x, y, z) &= (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)) \\ &= (2x, 2y, -2z)\end{aligned}$$

and its associated magnitude

$$\begin{aligned}|\nabla f(x, y, z)| &= \sqrt{(2x)^2 + (2y)^2 + (-2z)^2} \\ &= \sqrt{4x^2 + 4y^2 + 4z^2} \\ &= \sqrt{4(x^2 + y^2 + z^2)} \\ &= 2\sqrt{x^2 + y^2 + z^2}.\end{aligned}$$

So the unit normal vector at $(1, 1, -2)$ is

$$\begin{aligned}N(x, y, z) &= \frac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|} \\ &= \frac{(2x, 2y, -2z)}{2\sqrt{x^2 + y^2 + z^2}} \\ &= \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, -\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).\end{aligned}$$

Meanwhile, we also observe that, on the hyperboloid of revolution $x^2 + y^2 - z^2 = 1$, we have $z^2 = x^2 + y^2 - 1$, or equivalently $z = \pm\sqrt{x^2 + y^2 - 1}$. So we can consider the third coordinate of $N(x, y, z)$, which we write

$$\begin{aligned}N_z(x, y, z) &:= -\frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ &= -\frac{\pm\sqrt{x^2 + y^2 - 1}}{\sqrt{x^2 + y^2 + (x^2 + y^2 - 1)}} \\ &= \frac{\mp\sqrt{x^2 + y^2 - 1}}{\sqrt{2(x^2 + y^2) - 1}} \\ &= \mp\sqrt{\frac{x^2 + y^2 - 1}{2(x^2 + y^2) - 1}}.\end{aligned}$$

So we just found out that the third coordinate N_z is *not* be a well-defined function since the $\mp$ sign in the expression of N_z signifies that N_z takes on two simultaneous values. Our only workaround for this is that we can split N_z into two components

$$N_z^+(x, y, z) := \sqrt{\frac{x^2 + y^2 - 1}{2(x^2 + y^2) - 1}}$$

and

$$N_z^-(x, y, z) := -\sqrt{\frac{x^2 + y^2 - 1}{2(x^2 + y^2) - 1}},$$

both of which are functions (whereas N_z itself is not). We must invoke polar coordinates by writing $r^2 = u^2 + v^2$ for some $r > 0$; this allows us to rewrite our z-coordinate of N^+ as a function of r only: $N_z = \sqrt{\frac{r^2-1}{2r^2-1}}$. An important observation here is that, on the hyperboloid of revolution described by the equation $x^2 + y^2 - z^2 = 1$, we have

$$\begin{aligned}r^2 &= x^2 + y^2 \\ &= (x^2 + y^2 - z^2) + z^2 \\ &= 1 + z^2 \\ &\geq 1,\end{aligned}$$

which means in particular that we can worry about the z-coordinate $N_z^+(r) = \mp\sqrt{\frac{r^2-1}{2r^2-1}}$ only for all values $r > 0$ that satisfy $r^2 \geq 1$, or equivalently for all $r \geq 1$. Now, we will observe the behavior of N_z as a function of r on the interval $[1, \infty)$. First, note that, at $r = 1$, we have

$$\begin{aligned}N_z^+(1) &= \sqrt{\frac{(1)^2 - 1}{2(1)^2 - 1}} \\ &= 0.\end{aligned}$$

Next, we observe that $N_z^+(r)$ is increasing for all $r \in [1, \infty)$. The reason is that, for all $r, \tilde{r} \in [1, \infty)$ satisfying $r \leq \tilde{r}$, then we have $r^2 \leq \tilde{r}^2$ (since $r \geq 1$ and $\tilde{r} \geq 1$), and so

$$\begin{aligned}(r^2 - 1)(2\tilde{r}^2 - 1) &= 2r^2\tilde{r}^2 - r^2 - 2\tilde{r}^2 + 1 \\ &\leq 2r^2\tilde{r}^2 - \tilde{r}^2 - 2r^2 + 1 \\ &= (\tilde{r}^2 - 1)(2r^2 - 1),\end{aligned}$$

which algebraically implies $\frac{r^2-1}{2r^2-1} \leq \frac{\tilde{r}^2-1}{2\tilde{r}^2-1}$, and so

$$\begin{aligned}N_z^+(r) &= \sqrt{\frac{r^2 - 1}{2r^2 - 1}} \\ &\leq \sqrt{\frac{\tilde{r}^2 - 1}{2\tilde{r}^2 - 1}} \\ &= N_z^+(\tilde{r})\end{aligned}$$

This completes the proof that $N_z^+(r)$ is increasing. Finally, we have the limit

$$\begin{aligned}\lim_{r\to\infty} N_z^+(r) &= \lim_{r\to\infty} \sqrt{\frac{r^2 - 1}{2r^2 - 1}} \\ &= \lim_{r\to\infty} \sqrt{\frac{1 - \frac{1}{r^2}}{2 - \frac{1}{r^2}}} \\ &= \sqrt{\frac{1 - 0}{2 - 0}} \\ &= \frac{1}{\sqrt{2}}.\end{aligned}$$

We will work with the z-coordinate $N_z^- := -\sqrt{\frac{u^2+v^2-1}{2(u^2+v^2)-1}}$ of the normal vector N^-. We must invoke polar coordinates by writing $r^2 = u^2 + v^2$ for some $r > 0$; this allows us to rewrite our z-coordinate of N^- as a function of r only: $N_z^- = \sqrt{\frac{r^2-1}{2r^2-1}}$. An important observation here is that, on the hyperboloid of revolution described by the equation $x^2 + y^2 - z^2 = 1$, we have

$$\begin{aligned}r^2 &= u^2 + v^2 \\ &= x^2 + y^2 \\ &= (x^2 + y^2 - z^2) + z^2 \\ &= 1 + z^2 \\ &\geq 1,\end{aligned}$$

which means in particular that we can worry about the z-coordinate $N_z^-(r) = \sqrt{\frac{r^2-1}{2r^2-1}}$ only for all values $r > 0$ that satisfy $r^2 \geq 1$, or equivalently for all $r \geq 1$. Now, we will observe the behavior of N_z^+ as a function of r on the interval $[1, \infty)$. First, note that, at $r = 1$, we have

$$\begin{aligned}N_z^-(1) &= -\sqrt{\frac{(1)^2 - 1}{2(1)^2 - 1}} \\ &= 0.\end{aligned}$$

Next, we observe that $N_z^-(r)$ is decreasing for all $r \in [1, \infty)$. The reason is that, for all $r, \tilde{r} \in [1, \infty)$ satisfying $r \leq \tilde{r}$, then we have $r^2 \leq \tilde{r}^2$ (since $r \geq 1$ and $\tilde{r} \geq 1$), and so

$$\begin{aligned}(r^2 - 1)(2\tilde{r}^2 - 1) &= 2r^2\tilde{r}^2 - r^2 - 2\tilde{r}^2 + 1 \\ &\leq 2r^2\tilde{r}^2 - \tilde{r}^2 - 2r^2 + 1 \\ &= (\tilde{r}^2 - 1)(2r^2 - 1),\end{aligned}$$

which algebraically implies $\frac{r^2-1}{2r^2-1} \geq \frac{\tilde{r}^2-1}{2\tilde{r}^2-1}$, and so

$$\begin{aligned}N_z^-(r) &= -\sqrt{\frac{r^2 - 1}{2r^2 - 1}} \\ &\leq -\sqrt{\frac{\tilde{r}^2 - 1}{2\tilde{r}^2 - 1}} \\ &= N_z^-(\tilde{r}),\end{aligned}$$

or $N_z^-(r) \geq N_z^-(\tilde{r})$. This completes the proof that $N_z^-(r)$ is decreasing. Finally, we have the limit

$$\begin{aligned}\lim_{r\to\infty} N_z^-(r) &= \lim_{r\to\infty}\left(-\sqrt{\frac{r^2-1}{2r^2-1}}\right)\\ &= -\lim_{r\to\infty}\sqrt{\frac{1-\frac{1}{r^2}}{2-\frac{1}{r^2}}}\\ &= -\sqrt{\frac{1-0}{2-0}}\\ &= -\frac{1}{\sqrt{2}}.\end{aligned}$$

Hence, for all $r \geq 1$, we have $0 \geq N_z^-(r) > -\frac{1}{\sqrt{2}}$. Since we also already said much earlier $0 \leq N_z^+(r) < \frac{1}{\sqrt{2}}$, we can combine N^+, N^- together to conclude that $-\frac{1}{\sqrt{2}} < N_z(r) < \frac{1}{\sqrt{2}}$, or equivalently $|N_z(r)| < \frac{1}{\sqrt{2}}$, for all $r \geq 1$. Hence, the image of the unit normal vector N in S^2 is contained in the equatorial belt $T := \{(x, y, z) \in S^2 \mid |z| < \frac{1}{\sqrt{2}}\}$. Meanwhile, as in part a, we also observe that the first two coordinates $N_x := -\frac{x}{\sqrt{x^2+y^2+z^2}}$ and $N_y := -\frac{y}{\sqrt{x^2+y^2+z^2}}$ of N are arbitrary real numbers depending on $(x, y) \in \mathbb{R}^2$; the significance of this fact is that we can conclude that the image of N is not part of T (that is, properly contained in some strict subset of T), but rather the image of N is actually equal to T itself. □

Solution 2: Parametrization, as done in part a

Proof. Solving for z from $x^2 + y^2 - z^2 = 1$, we get two functions $z = \pm\sqrt{x^2+y^2-1}$, which allow us to consider their corresponding parametrizations $\mathbf{x}^+, \mathbf{x}^- : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\mathbf{x}^+(u, v) := (u, v, \sqrt{u^2+v^2-1})$ and $\mathbf{x}^-(u, v) := (u, v, -\sqrt{u^2+v^2-1})$. Let us work with the first parametrization $\mathbf{x}^+$ first. The partial derivatives of $\mathbf{x}^+$ are

$$\begin{aligned}\mathbf{x}_u^+(u, v) &= \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial}{\partial u}(u, v, \sqrt{u^2+v^2-1})\\ &= \left(\frac{\partial}{\partial u}(u), \frac{\partial}{\partial u}(v), \frac{\partial}{\partial u}(\sqrt{u^2+v^2-1})\right)\\ &= \left(1, 0, \frac{u}{\sqrt{u^2+v^2-1}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_v^+(u, v) &= \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial}{\partial v}(u, v, \sqrt{u^2+v^2-1})\\ &= \left(\frac{\partial}{\partial v}(u), \frac{\partial}{\partial v}(v), \frac{\partial}{\partial v}(\sqrt{u^2+v^2-1})\right)\\ &= \left(1, 0, \frac{v}{\sqrt{u^2+v^2-1}}\right).\end{aligned}$$

Then we obtain the cross product

$$\begin{aligned}\mathbf{x}_u^+(u, v) \times \mathbf{x}_v^+(u, v) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2-1}}\\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2-1}}\end{vmatrix}\\ &= \begin{vmatrix}0 & \frac{v}{\sqrt{u^2+v^2-1}}\\ 1 & \frac{v}{\sqrt{u^2+v^2-1}}\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & \frac{v}{\sqrt{u^2+v^2-1}}\\ 0 & \frac{v}{\sqrt{u^2+v^2-1}}\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\mathbf{k}\\ &= \left((0)\left(\frac{v}{\sqrt{u^2+v^2-1}}\right) - (1)\left(\frac{u}{\sqrt{u^2+v^2-1}}\right)\right)\mathbf{i}\\ &\quad - \left((1)\left(\frac{v}{\sqrt{u^2+v^2-1}}\right) - (0)\left(\frac{u}{\sqrt{u^2+v^2-1}}\right)\right)\mathbf{j} + ((1)(0) - (0)(1))\mathbf{k}\\ &= \left(-\frac{u}{\sqrt{u^2+v^2-1}}\right)\mathbf{i} + \left(-\frac{v}{\sqrt{u^2+v^2-1}}\right)\mathbf{j} + (1)\mathbf{k}\\ &= \left(-\frac{u}{\sqrt{u^2+v^2-1}}, -\frac{v}{\sqrt{u^2+v^2-1}}, 1\right)\end{aligned}$$

and its associated magnitude

$$\begin{aligned}
|\mathbf{x}_u^+(u,v) \times \mathbf{x}_v^+(u,v)| &= \sqrt{\left(-\frac{u}{\sqrt{u^2+v^2-1}}\right)^2 + \left(-\frac{v}{\sqrt{u^2+v^2-1}}\right)^2 + (1)^2} \\
&= \sqrt{\frac{u^2}{u^2+v^2-1} + \frac{v^2}{u^2+v^2-1} + 1} \\
&= \sqrt{\frac{u^2}{u^2+v^2-1} + \frac{v^2}{u^2+v^2-1} + \frac{u^2+v^2-1}{u^2+v^2-1}} \\
&= \sqrt{\frac{u^2+v^2+(u^2+v^2-1)}{u^2+v^2-1}} \\
&= \sqrt{\frac{2(u^2+v^2)-1}{u^2+v^2-1}}.
\end{aligned}$$

Thus, according to Definition 1 of Section 3-2 (c.f. page 136 of do Carmo), the normal vector $N^+ : S \to S^2$, where $S \subset \mathbb{R}^3$ is a surface and $S^2 := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ is a sphere, is given by

$$\begin{aligned}
N^+(u,v) &= \frac{\mathbf{x}_u^+(u,v) \times \mathbf{x}_v^+(u,v)}{|\mathbf{x}_u^+(u,v) \times \mathbf{x}_v^+(u,v)|} \\
&= \frac{\left(-\frac{u}{\sqrt{u^2+v^2-1}}, -\frac{v}{\sqrt{u^2+v^2-1}}, 1\right)}{\sqrt{\frac{2(u^2+v^2)-1}{u^2+v^2-1}}} \\
&= \frac{\left(-\frac{u}{\sqrt{u^2+v^2-1}}, -\frac{v}{\sqrt{u^2+v^2-1}}, 1\right)}{\sqrt{\frac{2(u^2+v^2)-1}{u^2+v^2-1}}} \frac{\sqrt{u^2+v^2-1}}{\sqrt{u^2+v^2-1}} \\
&= \frac{(-u, -v, \sqrt{u^2+v^2-1})}{\sqrt{2(u^2+v^2)-1}} \\
&= \left(-\frac{u}{\sqrt{2(u^2+v^2)-1}}, -\frac{v}{\sqrt{2(u^2+v^2)-1}}, \sqrt{\frac{u^2+v^2-1}{2(u^2+v^2)-1}}\right).
\end{aligned}$$

Much like in part a, we will work with the z-coordinate $N_z^+ := \sqrt{\frac{u^2+v^2-1}{2(u^2+v^2)-1}}$ of the normal vector N^+. We must invoke polar coordinates by writing $r^2 = u^2+v^2$ for some $r > 0$; this allows us to rewrite our z-coordinate of N^+ as a function of r only: $N_z^+ = \sqrt{\frac{r^2-1}{2r^2-1}}$. An important observation here is that, on the hyperboloid of revolution described by the equation $x^2+y^2-z^2 = 1$, we have

$$\begin{aligned}
r^2 &= u^2+v^2 \\
&= x^2+y^2 \\
&= (x^2+y^2-z^2)+z^2 \\
&= 1+z^2 \\
&\geq 1,
\end{aligned}$$

which means in particular that we can worry about the z-coordinate $N_z^+(r) = \sqrt{\frac{r^2-1}{2r^2-1}}$ only for all values $r > 0$ that satisfy $r^2 \geq 1$, or equivalently for all $r \geq 1$. Now, we will observe the behavior of N_z^+ as a function of r on the interval $[1,\infty)$. First, note that, at $r = 1$, we have

$$\begin{aligned}
N_z^+(1) &= \sqrt{\frac{(1)^2-1}{2(1)^2-1}} \\
&= 0.
\end{aligned}$$

Next, we observe that $N_z^+(r)$ is increasing for all $r \in [1,\infty)$. The reason is that, for all $r, \tilde{r} \in [1,\infty)$ satisfying $r \leq \tilde{r}$, then we have $r^2 \leq \tilde{r}^2$ (since $r \geq 1$ and $\tilde{r} \geq 1$), and so

$$\begin{aligned}
(r^2-1)(2\tilde{r}^2-1) &= 2r^2\tilde{r}^2 - r^2 - 2\tilde{r}^2 + 1 \\
&\leq 2r^2\tilde{r}^2 - \tilde{r}^2 - 2r^2 + 1 \\
&= (\tilde{r}^2-1)(2r^2-1),
\end{aligned}$$

which algebraically implies $\frac{r^2-1}{2r^2-1} \leq \frac{\tilde{r}^2-1}{2\tilde{r}^2-1}$, and so

$$\begin{aligned} N_z^+(r) &= \sqrt{\frac{r^2-1}{2r^2-1}} \\ &\leq \sqrt{\frac{\tilde{r}^2-1}{2\tilde{r}^2-1}} \\ &= N_z^+(\tilde{r}) \end{aligned}$$

This completes the proof that $-N_z^-(r)$ is decreasing. Finally, we have the limit

$$\begin{aligned} \lim_{r\to\infty} N_z^+(r) &= \lim_{r\to\infty} \sqrt{\frac{r^2-1}{2r^2-1}} \\ &= \lim_{r\to\infty} \sqrt{\frac{1-\frac{1}{r^2}}{2-\frac{1}{r^2}}} \\ &= \sqrt{\frac{1-0}{2-0}} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Hence, for all $r \geq 1$, we have $0 \leq N_z^+(r) < \frac{1}{\sqrt{2}}$. This means that the image of N^+ must be contained in $T^+ := \{(x,y,z) \in S^2 \mid 0 \leq z < \frac{1}{\sqrt{2}}\}$. Meanwhile, as in part a, we also observe that the first two coordinates $N_x^+ := -\frac{u}{\sqrt{2(u^2+v^2)-1}}$ and $N_y^+ := -\frac{v}{\sqrt{2(u^2+v^2)-1}}$ of N^+ are arbitrary real numbers depending on $(u,v) \in \mathbb{R}^2$; the significance of this fact is that we can conclude that the image of N^+ is not part of T^+ (that is, properly contained in some strict subset of T^+), but rather the image of N is actually equal to T^+ itself.

At this point, our work with $\mathbf{x}^+$ is all done; we will now work with the other parametrization $\mathbf{x}^-$. The partial derivatives of $\mathbf{x}^-$ are

$$\begin{aligned} \mathbf{x}_u^-(u,v) &= \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial}{\partial u}(u, v, -\sqrt{u^2+v^2-1}) \\ &= \left(\frac{\partial}{\partial u}(u), \frac{\partial}{\partial u}(v), \frac{\partial}{\partial u}(-\sqrt{u^2+v^2-1})\right) \\ &= \left(1, 0, -\frac{u}{\sqrt{u^2+v^2-1}}\right) \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_v^-(u,v) &= \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial}{\partial v}(u, v, -\sqrt{u^2+v^2-1}) \\ &= \left(\frac{\partial}{\partial v}(u), \frac{\partial}{\partial v}(v), \frac{\partial}{\partial v}(-\sqrt{u^2+v^2-1})\right) \\ &= \left(1, 0, -\frac{v}{\sqrt{u^2+v^2-1}}\right). \end{aligned}$$

Then we obtain the cross product

$$\begin{aligned} \mathbf{x}_u^-(u,v) \times \mathbf{x}_v^-(u,v) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{u}{\sqrt{u^2+v^2-1}} \\ 0 & 1 & -\frac{v}{\sqrt{u^2+v^2-1}} \end{vmatrix} \\ &= \begin{vmatrix} 0 & -\frac{v}{\sqrt{u^2+v^2-1}} \\ 1 & -\frac{v}{\sqrt{u^2+v^2-1}} \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -\frac{v}{\sqrt{u^2+v^2-1}} \\ 0 & -\frac{v}{\sqrt{u^2+v^2-1}} \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= \left((0)\left(-\frac{v}{\sqrt{u^2+v^2-1}}\right) - (1)\left(-\frac{u}{\sqrt{u^2+v^2-1}}\right)\right)\mathbf{i} \\ &\quad - \left((1)\left(-\frac{v}{\sqrt{u^2+v^2-1}}\right) - (0)\left(-\frac{u}{\sqrt{u^2+v^2-1}}\right)\right)\mathbf{j} + ((1)(0)-(0)(1))\mathbf{k} \\ &= \left(\frac{u}{\sqrt{u^2+v^2-1}}\right)\mathbf{i} + \left(\frac{v}{\sqrt{u^2+v^2-1}}\right)\mathbf{j} + (1)\mathbf{k} \\ &= \left(\frac{u}{\sqrt{u^2+v^2-1}}, \frac{v}{\sqrt{u^2+v^2-1}}, 1\right) \end{aligned}$$

and its associated magnitude

$$
\begin{aligned}
|\mathbf{x}_u^-(u,v) \times \mathbf{x}_v^-(u,v)| &= \sqrt{\left(\frac{u}{\sqrt{u^2+v^2-1}}\right)^2 + \left(\frac{v}{\sqrt{u^2+v^2-1}}\right)^2 + (1)^2} \\
&= \sqrt{\frac{u^2}{u^2+v^2-1} + \frac{v^2}{u^2+v^2-1} + 1} \\
&= \sqrt{\frac{u^2}{u^2+v^2-1} + \frac{v^2}{u^2+v^2-1} + \frac{u^2+v^2-1}{u^2+v^2-1}} \\
&= \sqrt{\frac{u^2+v^2+(u^2+v^2-1)}{u^2+v^2-1}} \\
&= \sqrt{\frac{2(u^2+v^2)-1}{u^2+v^2-1}}.
\end{aligned}
$$

Thus, according to Definition 1 of Section 3-2 (c.f. page 136 of do Carmo), the normal vector $N^- : S \to S^2$, where $S \subset \mathbb{R}^3$ is a surface and $S^2 := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ is a sphere, is given by

$$
\begin{aligned}
N^-(u,v) &= \frac{\mathbf{x}_u^-(u,v) \times \mathbf{x}_v^-(u,v)}{|\mathbf{x}_u^-(u,v) \times \mathbf{x}_v^-(u,v)|} \\
&= \frac{\left(\frac{u}{\sqrt{u^2+v^2-1}}, \frac{v}{\sqrt{u^2+v^2-1}}, 1\right)}{\sqrt{\frac{2(u^2+v^2)-1}{u^2+v^2-1}}} \\
&= \frac{\left(\frac{u}{\sqrt{u^2+v^2-1}}, \frac{v}{\sqrt{u^2+v^2-1}}, 1\right)}{\sqrt{\frac{2(u^2+v^2)-1}{u^2+v^2-1}}} \frac{\sqrt{u^2+v^2-1}}{\sqrt{u^2+v^2-1}} \\
&= \frac{(u, v, \sqrt{u^2+v^2-1})}{\sqrt{2(u^2+v^2)-1}} \\
&= \left(\frac{u}{\sqrt{2(u^2+v^2)-1}}, \frac{v}{\sqrt{2(u^2+v^2)-1}}, \sqrt{\frac{u^2+v^2-1}{2(u^2+v^2)-1}}\right).
\end{aligned}
$$

However, the negative signs appear on the first two coordinates in N^-, unlike in N^+ which has a negative sign only in the third coordinate. In other words, N^- is an inward-pointing vector (whereas N^+ is not), but this means that $-N^-$ is an outward-pointing vector, and we will work this $-N^-$ instead for the rest of this. To this end, we can write the expression of $-N^-$ as

$$
-N^-(u,v) = \left(-\frac{u}{\sqrt{2(u^2+v^2)-1}}, -\frac{v}{\sqrt{2(u^2+v^2)-1}}, -\sqrt{\frac{u^2+v^2-1}{2(u^2+v^2)-1}}\right).
$$

Much like in part a, we will work with the z-coordinate $-N_z^- := -\sqrt{\frac{u^2+v^2-1}{2(u^2+v^2)-1}}$ of the normal vector $-N^-$. We must invoke polar coordinates by writing $r^2 = u^2+v^2$ for some $r > 0$; this allows us to rewrite our z-coordinate of N^+ as a function of r only: $-N_z^- = \sqrt{\frac{r^2-1}{2r^2-1}}$. An important observation here is that, on the hyperboloid of revolution described by the equation $x^2+y^2-z^2 = 1$, we have

$$
\begin{aligned}
r^2 &= u^2+v^2 \\
&= x^2+y^2 \\
&= (x^2+y^2-z^2)+z^2 \\
&= 1+z^2 \\
&\geq 1,
\end{aligned}
$$

which means in particular that we can worry about the z-coordinate $-N_z^-(r) = \sqrt{\frac{r^2-1}{2r^2-1}}$ only for all values $r > 0$ that satisfy $r^2 \geq 1$, or equivalently for all $r \geq 1$. Now, we will observe the behavior of N_z^+ as a function of r on the interval $[1,\infty)$. First, note that, at $r = 1$, we have

$$
\begin{aligned}
-N_z^-(1) &= -\sqrt{\frac{(1)^2-1}{2(1)^2-1}} \\
&= 0.
\end{aligned}
$$

Next, we observe that $-N_z^-(r)$ is decreasing for all $r \in [1,\infty)$. The reason is that, for all $r, \tilde{r} \in [1,\infty)$ satisfying $r \leq \tilde{r}$, then we have $r^2 \leq \tilde{r}^2$ (since $r \geq 1$ and $\tilde{r} \geq 1$), and so

$$\begin{aligned}(r^2-1)(2\tilde{r}^2-1) &= 2r^2\tilde{r}^2 - r^2 - 2\tilde{r}^2 + 1\\ &\leq 2r^2\tilde{r}^2 - \tilde{r}^2 - 2r^2 + 1\\ &= (\tilde{r}^2-1)(2r^2-1),\end{aligned}$$

which algebraically implies $\frac{r^2-1}{2r^2-1} \geq \frac{\tilde{r}^2-1}{2\tilde{r}^2-1}$, and so

$$\begin{aligned}N_z^-(r) &= \sqrt{\frac{r^2-1}{2r^2-1}}\\ &\leq \sqrt{\frac{\tilde{r}^2-1}{2\tilde{r}^2-1}}\\ &= N_z^-(\tilde{r}),\end{aligned}$$

or $-N_z^-(r) \geq -N_z^-(\tilde{r})$. This completes the proof that $-N_z^-(r)$ is decreasing. Finally, we have the limit

$$\begin{aligned}\lim_{r\to\infty}(-N_z^-(r)) &= -\lim_{r\to\infty}\sqrt{\frac{r^2-1}{2r^2-1}}\\ &= -\lim_{r\to\infty}\sqrt{\frac{1-\frac{1}{r^2}}{2-\frac{1}{r^2}}}\\ &= -\sqrt{\frac{1-0}{2-0}}\\ &= -\frac{1}{\sqrt{2}}.\end{aligned}$$

Hence, for all $r \geq 1$, we have $0 \geq N_z^-(r) > -\frac{1}{\sqrt{2}}$. This means that the image of N^+ must be contained in $T^- := \{(x, y, z) \in S^2 \mid 0 \geq z > \frac{1}{\sqrt{2}}\}$. Meanwhile, as in part a, we also observe that the first two coordinates $-N_x^- := -\frac{u}{\sqrt{2(u^2+v^2)-1}}$ and $-N_y^- := -\frac{v}{\sqrt{2(u^2+v^2)-1}}$ of $-N^-$ are arbitrary real numbers depending on $(u, v) \in \mathbb{R}^2$; the significance of this fact is that we can conclude that the image of $-N^-$ is not part of T^+ (that is, properly contained in some strict subset of T^-), but rather the image of N is actually equal to T^- itself.

Finally, we will now consider our two parametrizations $\mathbf{x}^\pm$ simultaneously. We just established in our last paragraph that $\mathbf{x}^-$ is a parametrization that induces the unit normal vector $-N^-$ whose image is $T^- \subset S^2$. Likewise, we also already established in two paragraphs above that $\mathbf{x}^+$ is a parametrization that induces the unit normal vector N^+ whose image is $T^+ \subset S^2$. Combining these two results, we conclude that the two parametrizations $\mathbf{x}^\pm$ simultaneously establish the unit normal vectors $\pm N^\pm$, whose combined image in S^2 is $T^+ \cup T^- = \{(x, y, z) \in S^2 \mid -\frac{1}{\sqrt{2}} < z < \frac{1}{\sqrt{2}}\}$. □