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Solutions to assigned homework problems from *Differential Geometry of Curves and Surfaces* by Manfredo Perdigão do Carmo

Assignment 7 – pages 168-172: 5abc,16 and page 212: 12

3-3.5. Consider the parametrized surface

$$\mathbf{x}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\right)$$

and show that

a. The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

Proof. We obtain the first derivatives

$$\begin{aligned}
\mathbf{x}_u(u,v) &= \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial}{\partial u}\left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\right) \\
&= \left(\frac{\partial}{\partial u}\left(u - \frac{u^3}{3} + uv^2\right), \frac{\partial}{\partial u}\left(v - \frac{v^3}{3} + u^2 v\right), \frac{\partial}{\partial u}(u^2 - v^2)\right) \\
&= (1 - u^2 + v^2, 2uv, 2u)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{x}_v(u,v) &= \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial}{\partial v}\left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\right) \\
&= \left(\frac{\partial}{\partial v}\left(u - \frac{u^3}{3} + uv^2\right), \frac{\partial}{\partial v}\left(v - \frac{v^3}{3} + u^2 v\right), \frac{\partial}{\partial v}(u^2 - v^2)\right) \\
&= (2uv, 1 - v^2 + u^2, -2v)
\end{aligned}$$

So the coefficients of the first fundamental form are

$$\begin{aligned}
E(u,v) &= \mathbf{x}_u(u,v) \cdot \mathbf{x}_u(u,v) \\
&= (1 - u^2 + v^2, 2uv, 2u) \cdot (1 - u^2 + v^2, 2uv, 2u) \\
&= (1 - u^2 + v^2)(1 - u^2 + v^2) + (2uv)(2uv) + (2u)(2u) \\
&= (1 - 2u^2 + 2v^2 - 2u^2v^2 + u^4 + v^4) + 4u^2v^2 + 4u^2 \\
&= 1 + 2u^2 + 2v^2 + 2u^2v^2 + u^4 + v^4 \\
&= (1 + u^2 + v^2)(1 + u^2 + v^2) \\
&= (1 + u^2 + v^2)^2,
\end{aligned}$$

as well as

$$\begin{aligned}
F(u,v) &= \mathbf{x}_u(u,v) \cdot \mathbf{x}_v(u,v) \\
&= (1 - u^2 + v^2, 2uv, 2u) \cdot (2uv, 1 - v^2 + u^2, -2v) \\
&= (1 - u^2 + v^2)(2uv) + (2uv)(1 - v^2 + u^2) + (2u)(-2v) \\
&= (2uv - 2u^3v + 2u^2v^3) + (2uv - 2uv^3 + 2u^3v) - 4uv \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
G(u,v) &= \mathbf{x}_v(u,v) \cdot \mathbf{x}_v(u,v) \\
&= (2uv, 1 - v^2 + u^2, -2v) \cdot (2uv, 1 - v^2 + u^2, -2v) \\
&= (2uv)(2uv) + (1 - v^2 + u^2)(1 - v^2 + u^2) + (-2v)(-2v) \\
&= 4u^2v^2 + (1 - 2v^2 + 2u^2 - 2u^2v^2 + v^4 + u^4) + 4v^2 \\
&= 1 + 2v^2 + 2u^2 + 2u^2v^2 + v^4 + u^4 \\
&= (1 + u^2 + v^2)(1 + u^2 + v^2) \\
&= (1 + u^2 + v^2)^2.
\end{aligned}$$

□

b. The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

Proof. We obtain the second derivatives

$$\begin{aligned}\mathbf{x}_{uu}(u,v) &= \frac{\partial \mathbf{x}_u}{\partial u} = \frac{\partial}{\partial u}(1-u^2+v^2, 2uv, 2u) \\ &= (-2u, 2v, 2),\end{aligned}$$

as well as

$$\begin{aligned}\mathbf{x}_{uv}(u,v) &= \frac{\partial \mathbf{x}_u}{\partial v} = \frac{\partial}{\partial v}(1-u^2+v^2, 2uv, 2u) \\ &= (2v, 2u, 0)\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_{vv}(u,v) &= \frac{\partial \mathbf{x}_v}{\partial v} = \frac{\partial}{\partial v}(2uv, 1-v^2+u^2, -2v) \\ &= (2u, -2v, -2).\end{aligned}$$

To obtain the normal vector N, first we recall the first derivatives $\mathbf{x}_u(u,v) = (1-u^2+v^2, 2uv, 2u)$ and $\mathbf{x}_v(u,v) = (2uv, 1-v^2+u^2, -2v)$. Then we obtain the cross product

$$\begin{aligned}\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1-u^2+v^2 & 2uv & 2u \\ 2uv & 1-v^2+u^2 & -2v\end{vmatrix} \\ &= \begin{vmatrix}2uv & 2u \\ 1-v^2+u^2 & -2v\end{vmatrix}\mathbf{i} - \begin{vmatrix}1-u^2+v^2 & 2u \\ 2uv & -2v\end{vmatrix}\mathbf{j} + \begin{vmatrix}1-u^2+v^2 & 2uv \\ 2uv & 1-v^2+u^2\end{vmatrix}\mathbf{k} \\ &= ((2uv)(-2v) - (1-v^2+u^2)(2u))\mathbf{i} - ((1-u^2+v^2)(-2v) - (2uv)(2u))\mathbf{j} \\ &\quad + ((1-u^2+v^2)(1-v^2+u^2) - (2uv)(2uv))\mathbf{k} \\ &= (-2uv^2 - 2u - 2u^3)\mathbf{i} + (2v + 2u^2v + 2v^3)\mathbf{j} + (1 - 2u^2v^2 - u^4 - v^4)\mathbf{k} \\ &= ((1+u^2+v^2)(-2u))\mathbf{i} + ((1+u^2+v^2)(2v))\mathbf{j} + ((1+u^2+v^2)(1-u^2-v^2))\mathbf{k} \\ &= (1+u^2+v^2)((-2u)\mathbf{i} + (2v)\mathbf{j} + (1-u^2-v^2)\mathbf{k}) \\ &= (1+u^2+v^2)(-2u, 2v, 1-u^2-v^2)\end{aligned}$$

and its associated magnitude

$$\begin{aligned}|\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)| &= (1+u^2+v^2)\sqrt{(-2u)^2+(2v)^2+(1-u^2-v^2)^2} \\ &= (1+u^2+v^2)\sqrt{4u^2+4v^2+(1-2u^2-2v^2+2u^2v^2+u^4+v^4)} \\ &= (1+u^2+v^2)\sqrt{1+2u^2+2v^2+2u^2v^2+u^4+v^4} \\ &= (1+u^2+v^2)\sqrt{(1+u^2+v^2)^2} \\ &= (1+u^2+v^2)(1+u^2+v^2) \\ &= (1+u^2+v^2)^2,\end{aligned}$$

and so the normal vector is

$$\begin{aligned}N(u,v) &= \frac{\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)}{|\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)|} \\ &= \frac{(1+u^2+v^2)(-2u, 2v, 1-u^2-v^2)}{(1+u^2+v^2)^2} \\ &= \frac{(1+u^2+v^2)(-2u, 2v, 1-u^2-v^2)}{(1+u^2+v^2)^2} \\ &= \frac{(-2u, 2v, 1-u^2-v^2)}{1+u^2+v^2}\end{aligned}$$

So the coefficients of the second fundamental form are

$$\begin{aligned}
e(u,v) &= N(u,v) \cdot \mathbf{x}_{uu}(u,v) \\
&= \frac{(-2u, 2v, 1-u^2-v^2)}{1+u^2+v^2} \cdot (-2u, 2v, 2) \\
&= \frac{(-2u)(-2u) + (2v)(2v) + (1-u^2-v^2)(2)}{1+u^2+v^2} \\
&= \frac{4u^2 + 4v^2 + (2 - 2u^2 - 2v^2)}{1+u^2+v^2} \\
&= \frac{2 + 2u^2 + 2v^2}{1+u^2+v^2} \\
&= \frac{2(1+u^2+v^2)}{1+u^2+v^2} \\
&= 2,
\end{aligned}$$

as well as

$$\begin{aligned}
f(u,v) &= N(u,v) \cdot \mathbf{x}_{uv}(u,v) \\
&= \frac{(-2u, 2v, 1-u^2-v^2)}{1+u^2+v^2} \cdot (2v, 2u, 0) \\
&= \frac{(-2u)(2v) + (2v)(2u) + (1-u^2-v^2)(0)}{1+u^2+v^2} \\
&= \frac{-4uv + 4uv + 0}{1+u^2+v^2} \\
&= \frac{0}{1+u^2+v^2} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
g(u,v) &= N(u,v) \cdot \mathbf{x}_{vv}(u,v) \\
&= \frac{(-2u, 2v, 1-u^2-v^2)}{1+u^2+v^2} \cdot (2u, -2v, -2) \\
&= \frac{(-2u)(2u) + (2v)(-2v) + (1-u^2-v^2)(-2)}{1+u^2+v^2} \\
&= \frac{-4u^2 - 4v^2 + (-2 + 2u^2 + 2v^2)}{1+u^2+v^2} \\
&= \frac{-2 - 2u^2 - 2v^2}{1+u^2+v^2} \\
&= \frac{-2(1+u^2+v^2)}{1+u^2+v^2} \\
&= -2,
\end{aligned}$$

as desired. □

c. The principal curvatures are

$$k_1 = \frac{2}{(1+u^2+v^2)^2}, \quad k_2 = -\frac{2}{(1+u^2+v^2)^2}.$$

Proof. According to page 162 of do Carmo, since from part a we have $F = f = 0$, the principal curvatures are

$$\begin{aligned}
k_1 &= \frac{e}{E} \\
&= \frac{2}{(1+u^2+v^2)^2}
\end{aligned}$$

and

$$\begin{aligned}
k_2 &= \frac{g}{G} \\
&= -\frac{2}{(1+u^2+v^2)^2},
\end{aligned}$$

as desired. □

3-3.16. Show that a surface which is compact (i.e. closed and bounded) in $\mathbb{R}^3$ has an elliptic point.

Proof. Let $p \in \mathbb{R}^3$ be an elliptic point, which means $\det(dN_p) > 0$ (c.f. do Carmo, page 146), where we recall that dN_p is the differential of the Gauss map N_p. Now, let S be a compact surface. Then there exists a sphere of a sufficiently large radius $R > 0$ such that S lies inside of the sphere, except at only one point—call it p—that touches the sphere. (Note: it would be helpful to draw a picture of this.) Let K_S and K_{S^2} denote respectively the Gaussian curvatures of the surface $S \subset \mathbb{R}^3$ and of the sphere $S^2 \subset \mathbb{R}^3$ at the point p. Then $K_{S^2} = \frac{1}{R^2} > 0$ for some large enough $R > 0$, where R is the radius of the sphere S^2. Also, $K_S \geq K_{S^2}$ at the point p, since S is contained inside S^2. Therefore,

$$\begin{aligned} \det(dN_p) &= K_S \\ &\geq K_{S^2} \\ &= \frac{1}{R^2} \\ &> 0, \end{aligned}$$

which means p is an elliptic point. □

3-5.12. Prove that there are no compact minimal surfaces in $\mathbb{R}^3$.

Proof. Suppose to the contrary that there exists some surface $S \subset \mathbb{R}^3$ that is both compact and minimal. Since S is miminal, its mean curvature vanishes everywhere (c.f. do Carmo, page 197), i.e. $H \equiv 0$. So we have

$$\begin{aligned} 0 &= H \\ &= \frac{k_1 + k_2}{2} \end{aligned}$$

which implies that k_1, k_2 have opposite signs. Consequently, we have

$$\begin{aligned} \det(dN_p) &= K \\ &= k_1 k_2 \\ &< 0 \end{aligned}$$

for any arbitrary point $p \in S$, which implies that S does not have any elliptic points. But this contradicts Exercise 3-3.16, which asserts that S has an elliptic point since we also assumed that S is compact. Therefore, no compact minimal surfaces exist in $\mathbb{R}^3$. □

Bun Wong: Compute the first and second fundamental forms, Gaussian curvature, mean curvature, and principal curvatures of the examples in Problems 1, 2, and 4 of Section 3.5 (on page 168 of do Carmo).

3-5.1. Hyperboloid $z = axy$

Proof. Consider the parametrized surface

$$\mathbf{x}(u, v) = (u, v, auv).$$

We obtain the first derivatives

$$\begin{aligned} \mathbf{x}_u(u, v) &= \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial}{\partial u}(u, v, auv) \\ &= \left(\frac{\partial}{\partial u}(u), \frac{\partial}{\partial u}(v), \frac{\partial}{\partial u}(auv) \right) \\ &= (1, 0, av) \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_v(u, v) &= \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial}{\partial v}(u, v, auv) \\ &= \left(\frac{\partial}{\partial v}(u), \frac{\partial}{\partial v}(v), \frac{\partial}{\partial v}(auv) \right) \\ &= (0, 1, au). \end{aligned}$$

So the coefficients of the first fundamental form are

$$\begin{aligned} E(u, v) &= \mathbf{x}_u(u, v) \cdot \mathbf{x}_u(u, v) \\ &= (1, 0, av) \cdot (1, 0, av) \\ &= (1)(1) + (0)(0) + (av)(av) \\ &= 1 + a^2 v^2 \end{aligned}$$

as well as

$$\begin{aligned} F(u,v) &= \mathbf{x}_u(u,v) \cdot \mathbf{x}_v(u,v) \\ &= (1,0,av) \cdot (0,1,au) \\ &= (1)(0) + (0)(1) + (av)(au) \\ &= a^2uv \end{aligned}$$

and

$$\begin{aligned} G(u,v) &= \mathbf{x}_v(u,v) \cdot \mathbf{x}_v(u,v) \\ &= (1,0,au) \cdot (1,0,au) \\ &= (1)(1) + (0)(0) + (au)(au) \\ &= 1 + a^2u^2. \end{aligned}$$

To find the coefficients of the second fundamental form, first we must compute the normal vector. The cross product of $\mathbf{x}_u(u,v)$ and $\mathbf{x}_v(u,v)$ is

$$\begin{aligned} \mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & av \\ 0 & 1 & au \end{vmatrix} \\ &= \begin{vmatrix} 0 & av \\ 1 & au \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & av \\ 0 & au \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= ((0)(au) - (1)(av))\mathbf{i} - ((1)(au) - (0)(av))\mathbf{j} + ((1)(1) - (0)(0))\mathbf{k} \\ &= (-av)\mathbf{i} + (-au)\mathbf{j} + (1)\mathbf{k} \\ &= (-av, -au, 1) \end{aligned}$$

and its associated magnitude

$$\begin{aligned} |\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)| &= \sqrt{(-av)^2 + (-au)^2 + (1)^2} \\ &= \sqrt{a^2v^2 + a^2u^2 + 1} \\ &= \sqrt{a^2(u^2 + v^2) + 1}, \end{aligned}$$

and so the normal vector is

$$\begin{aligned} N(u,v) &= \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)|} \\ &= \frac{(-av, -au, 1)}{\sqrt{a^2(u^2 + v^2) + 1}}. \end{aligned}$$

Meanwhile, we obtain the second derivatives

$$\begin{aligned} \mathbf{x}_{uu}(u,v) &= \frac{\partial \mathbf{x}_u}{\partial u} = \frac{\partial}{\partial u}(1,0,av) \\ &= \left(\frac{\partial}{\partial u}(1), \frac{\partial}{\partial u}(0), \frac{\partial}{\partial u}(av) \right) \\ &= (0,0,0), \end{aligned}$$

as well as

$$\begin{aligned} \mathbf{x}_{uv}(u,v) &= \frac{\partial \mathbf{x}_u}{\partial v} = \frac{\partial}{\partial v}(1,0,av) \\ &= \left(\frac{\partial}{\partial v}(1), \frac{\partial}{\partial v}(0), \frac{\partial}{\partial v}(av) \right) \\ &= (0,0,a) \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_{vv}(u,v) &= \frac{\partial \mathbf{x}_v}{\partial v} = \frac{\partial}{\partial v}(0,1,au) \\ &= \left(\frac{\partial}{\partial u}(0), \frac{\partial}{\partial u}(1), \frac{\partial}{\partial u}(au) \right) \\ &= (0,0,0). \end{aligned}$$

So the coefficients of the second fundamental form are

$$\begin{aligned}
e(u,v) &= N(u,v)\cdot \mathbf{x}_{uu}(u,v) \\
&= \frac{(-av,-au,1)}{\sqrt{a^2(u^2+v^2)+1}}\cdot(0,0,0) \\
&= \frac{(-av)(0)+(-au)(0)+(1)(0)}{\sqrt{a^2(u^2+v^2)+1}} \\
&= 0,
\end{aligned}$$

as well as

$$\begin{aligned}
f(u,v) &= N(u,v)\cdot \mathbf{x}_{uv}(u,v) \\
&= \frac{(-av,-au,1)}{\sqrt{a^2(u^2+v^2)+1}}\cdot(0,0,a) \\
&= \frac{(-av)(0)+(-au)(0)+(1)(a)}{\sqrt{a^2(u^2+v^2)+1}} \\
&= \frac{a}{\sqrt{a^2(u^2+v^2)+1}}
\end{aligned}$$

and

$$\begin{aligned}
g(u,v) &= N(u,v)\cdot \mathbf{x}_{vv}(u,v) \\
&= \frac{(-av,-au,1)}{\sqrt{a^2(u^2+v^2)+1}}\cdot(0,0,0) \\
&= \frac{(-av)(0)+(-au)(0)+(1)(0)}{\sqrt{a^2(u^2+v^2)+1}} \\
&= 0.
\end{aligned}$$

Using the formula on page 155 of do Carmo, the Gaussian curvature is

$$\begin{aligned}
K(u,v) &= \frac{eg-f^2}{EG-F^2} \\
&= \frac{(0)(0)-\left(\frac{a}{\sqrt{a^2(u^2+v^2)+1}}\right)^2}{(1+a^2v^2)(1+a^2u^2)-(a^2uv)^2} \\
&= \frac{0-\frac{a^2}{a^2(u^2+v^2)+1}}{(1+a^2(u^2+v^2)+a^4u^2v^2)-a^4u^2v^2} \\
&= -\frac{a^2}{(1+a^2(u^2+v^2))^2}.
\end{aligned}$$

Using the formula on page 156 of do Carmo, the mean curvature is

$$\begin{aligned}
H(u,v) &= \frac{1}{2}\frac{eG-2fF+gE}{EG-F^2} \\
&= \frac{1}{2}\frac{(0)(1+a^2u^2)-2\left(\frac{a}{\sqrt{a^2(u^2+v^2)+1}}\right)(a^2uv)+(0)(1+a^2v^2)}{(1+a^2v^2)(1+a^2u^2)-(a^2uv)^2} \\
&= \frac{1}{2}\frac{0-2\frac{a^3uv}{\sqrt{a^2(u^2+v^2)+1}}+0}{(1+a^2(u^2+v^2)+a^4u^2v^2)-a^4u^2v^2} \\
&= -\frac{a^3uv}{(1+a^2(u^2+v^2))^{\frac{3}{2}}}.
\end{aligned}$$

We will now compute the principal curvatures k_1, k_2. To do this, we will solve for k_1, k_2 from the formulas $H = \frac{k_1+k_2}{2}$ and $K = k_1k_2$. From $H = \frac{k_1+k_2}{2}$, we get $k_1 = 2H - k_2$, and so we get

$$\begin{aligned}
K &= k_1k_2 \\
&= (2H-k_2)k_2 \\
&= 2Hk_2 - k_2^2,
\end{aligned}$$

which is algebraically equivalent to the quadratic equation

$$k_2^2 - 2Hk_2 + K = 0.$$

Employing the quadratic formula, we get

$$\begin{aligned} k_2 &= \frac{-(-2H) \pm \sqrt{(-2H)^2 - 4(1)(K)}}{2(1)} \\ &= \frac{2H \pm \sqrt{4(H^2 - K)}}{2} \\ &= H \pm \sqrt{H^2 - K}. \end{aligned}$$

This also means

$$\begin{aligned} k_1 &= 2H - k_2 \\ &= 2H - (H \pm \sqrt{H^2 - K}) \\ &= H \mp \sqrt{H^2 - K}. \end{aligned}$$

As we conventionally require $k_1 > k_2$, we will choose $k_1 = H + \sqrt{H^2 - K}$ and $k_2 = H - \sqrt{H^2 - K}$. Substituting our expressions for H, K, our principal curvatures are

$$\begin{aligned} k_1 &= H + \sqrt{H^2 - K} \\ &= \left(-\frac{a^3 uv}{(1 + a^2(u^2 + v^2))^{\frac{3}{2}}} \right) + \sqrt{\left(-\frac{a^3 uv}{(1 + a^2(u^2 + v^2))^{\frac{3}{2}}} \right)^2 - \left(-\frac{a^2}{(1 + a^2(u^2 + v^2))^2} \right)} \\ &= -\frac{a^3 uv}{(1 + a^2(u^2 + v^2))^{\frac{3}{2}}} + \sqrt{\frac{a^6 u^2 v^2}{(1 + a^2(u^2 + v^2))^3} + \frac{a^2}{(1 + a^2(u^2 + v^2))^2}} \end{aligned}$$

and

$$\begin{aligned} k_2 &= H - \sqrt{H^2 - K} \\ &= \left(-\frac{a^3 uv}{(1 + a^2(u^2 + v^2))^{\frac{3}{2}}} \right) - \sqrt{\left(-\frac{a^3 uv}{(1 + a^2(u^2 + v^2))^{\frac{3}{2}}} \right)^2 - \left(-\frac{a^2}{(1 + a^2(u^2 + v^2))^2} \right)} \\ &= -\frac{a^3 uv}{(1 + a^2(u^2 + v^2))^{\frac{3}{2}}} - \sqrt{\frac{a^6 u^2 v^2}{(1 + a^2(u^2 + v^2))^3} + \frac{a^2}{(1 + a^2(u^2 + v^2))^2}} \end{aligned}$$

which are in terms of the variables a, u, v. □

3-5.2. Helicoid $x = v \cos u, y = v \sin u, z = cu$

Proof. Consider the parametrized surface

$$\mathbf{x}(u, v) = (v \cos u, v \sin u, cu).$$

We obtain the first derivatives

$$\begin{aligned} \mathbf{x}_u(u, v) &= \frac{\partial \mathbf{x}}{\partial u} = \frac{\partial}{\partial u}(v \cos u, v \sin u, cu) \\ &= \left(\frac{\partial}{\partial u}(v \cos u), \frac{\partial}{\partial u}(v \sin u), \frac{\partial}{\partial u}(cu) \right) \\ &= (-v \sin u, v \cos u, c) \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_v(u, v) &= \frac{\partial \mathbf{x}}{\partial v} = \frac{\partial}{\partial v}(v \cos u, v \sin u, cu) \\ &= \left(\frac{\partial}{\partial v}(v \cos u), \frac{\partial}{\partial v}(v \sin u), \frac{\partial}{\partial v}(cu) \right) \\ &= (\cos u, \sin u, 0) \end{aligned}$$

So the coefficients of the first fundamental form are

$$\begin{aligned} E(u, v) &= \mathbf{x}_u(u, v) \cdot \mathbf{x}_u(u, v) \\ &= (-v \sin u, v \cos u, c) \cdot (-v \sin u, v \cos u, c) \\ &= (-v \sin u)(-v \sin u) + (v \cos u)(v \cos u) + (c)(c) \\ &= v^2 \sin^2 u + v^2 \cos^2 u + c^2 \\ &= v^2 + c^2, \end{aligned}$$

as well as

$$\begin{aligned} F(u,v) &= \mathbf{x}_u(u,v)\cdot\mathbf{x}_v(u,v) \\ &= (-v\sin u, v\cos u, c)\cdot(\cos u,\sin u,0) \\ &= (-v\sin u)(\cos u)+(v\cos u)(\sin u)+(c)(0) \\ &= -v\sin u\cos u + v\sin u\cos u + 0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} G(u,v) &= \mathbf{x}_v(u,v)\cdot\mathbf{x}_v(u,v) \\ &= (\cos u,\sin u,0)\cdot(\cos u,\sin u,0) \\ &= (\cos u)(\cos u)+(\sin u)(\sin u)+(0)(0) \\ &= \cos^2 u+\sin^2 u+0 \\ &= 1. \end{aligned}$$

To find the coefficients of the second fundamental form, first we must compute the normal vector. The cross product of $\mathbf{x}_u(u,v)$ and $\mathbf{x}_v(u,v)$ is

$$\begin{aligned} \mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -v\sin u & v\cos u & c \\ \cos u & \sin u & 0 \end{vmatrix} \\ &= \begin{vmatrix} v\cos u & c \\ \sin u & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -v\sin u & c \\ \cos u & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -v\sin u & v\cos u \\ \cos u & \sin u \end{vmatrix}\mathbf{k} \\ &= ((v\cos u)(0)-(\sin u)(c))\mathbf{i} - ((-v\sin u)(0)-(\cos u)(c))\mathbf{j} + ((-v\sin u)(\sin u)-(\cos u)(v\cos u))\mathbf{k} \\ &= (-c\sin u)\mathbf{i}+(c\cos u)\mathbf{j}+(-v)\mathbf{k} \\ &= (-c\sin u, c\cos u, -v) \end{aligned}$$

and its associated magnitude

$$\begin{aligned} |\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)| &= \sqrt{(-c\sin u)^2+(c\cos u)^2+(-v)^2} \\ &= \sqrt{c^2\sin^2 u + c^2\cos^2 u + v^2} \\ &= \sqrt{c^2+v^2}, \end{aligned}$$

and so the normal vector is

$$\begin{aligned} N(u,v) &= \frac{\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)}{|\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)|} \\ &= \frac{(-c\sin u, c\cos u, -v)}{\sqrt{c^2+v^2}}. \end{aligned}$$

Meanwhile, we obtain the second derivatives

$$\begin{aligned} \mathbf{x}_{uu}(u,v) &= \frac{\partial \mathbf{x}_u}{\partial u} = \frac{\partial}{\partial u}(-v\sin u, v\cos u, c) \\ &= \left(\frac{\partial}{\partial u}(-v\sin u), \frac{\partial}{\partial u}(v\cos u), \frac{\partial}{\partial u}(c)\right) \\ &= (-v\cos u, -v\sin u, 0), \end{aligned}$$

as well as

$$\begin{aligned} \mathbf{x}_{uv}(u,v) &= \frac{\partial \mathbf{x}_u}{\partial v} = \frac{\partial}{\partial v}(-v\sin u, v\cos u, c) \\ &= \left(\frac{\partial}{\partial v}(-v\sin u), \frac{\partial}{\partial v}(v\cos u), \frac{\partial}{\partial v}(c)\right) \\ &= (-\sin u, \cos u, 0) \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_{vv}(u,v) &= \frac{\partial \mathbf{x}_v}{\partial v} = \frac{\partial}{\partial v}(\cos u, \sin u, 0) \\ &= \left(\frac{\partial}{\partial v}(\cos u), \frac{\partial}{\partial v}(\sin u), \frac{\partial}{\partial v}(0)\right) \\ &= (0,0,0). \end{aligned}$$

So the coefficients of the second fundamental form are

$$\begin{aligned} e(u,v) &= N(u,v)\cdot \mathbf{x}_{uu}(u,v) \\ &= \frac{(-c\sin u, c\cos u, -v)}{\sqrt{c^2+v^2}}\cdot(-v\cos u, -v\sin u, 0) \\ &= \frac{(-c\sin u)(-v\cos u)+(c\cos u)(-v\sin u)+(-v)(0)}{\sqrt{c^2+v^2}} \\ &= \frac{cv\sin u\cos u - cv\sin u\cos u + 0}{\sqrt{c^2+v^2}} \\ &= 0, \end{aligned}$$

as well as

$$\begin{aligned} f(u,v) &= N(u,v)\cdot \mathbf{x}_{uv}(u,v) \\ &= \frac{(-c\sin u, c\cos u, -v)}{\sqrt{c^2+v^2}}\cdot(-\sin u, \cos u, 0) \\ &= \frac{(-c\sin u)(-\sin u)+(c\cos u)(\cos u)+(-v)(0)}{\sqrt{c^2+v^2}} \\ &= \frac{c\sin^2 u + c\cos^2 u + 0}{\sqrt{c^2+v^2}} \\ &= \frac{c}{c^2+v^2} \end{aligned}$$

and

$$\begin{aligned} g(u,v) &= N(u,v)\cdot \mathbf{x}_{vv}(u,v) \\ &= \frac{(-c\sin u, c\cos u, -v)}{\sqrt{c^2+v^2}}\cdot(0,0,0) \\ &= \frac{(-c\sin u)(0)+(c\cos u)(0)+(-v)(0)}{\sqrt{c^2+v^2}} \\ &= \frac{0}{\sqrt{c^2+v^2}} \\ &= 0. \end{aligned}$$

Using the formula on page 155 of do Carmo, the Gaussian curvature is

$$\begin{aligned} K(u,v) &= \frac{eg-f^2}{EG-F^2} \\ &= \frac{(0)(0)-(\frac{c}{\sqrt{c^2+v^2}})^2}{(v^2+c^2)(1)-(0)^2} \\ &= -\frac{c^2}{(v^2+c^2)^2}. \end{aligned}$$

Using the formula on page 156 of do Carmo, the mean curvature is

$$\begin{aligned} H(u,v) &= \frac{1}{2}\frac{eG-2fF+gE}{EG-F^2} \\ &= \frac{1}{2}\frac{(0)(1)-2(\frac{c}{\sqrt{c^2+v^2}})(0)+(0)(v^2+c^2)}{(v^2+c^2)(1)-(0)^2} \\ &= 0. \end{aligned}$$

We will now compute the principal curvatures k_1, k_2. To do this, we will solve for k_1, k_2 from the formulas $H = \frac{k_1+k_2}{2}$ and $K = k_1k_2$. From $H = \frac{k_1+k_2}{2}$, we get $k_1 = 2H - k_2$, and so we get

$$\begin{aligned} K &= k_1k_2 \\ &= (2H-k_2)k_2 \\ &= 2Hk_2 - k_2^2, \end{aligned}$$

which is algebraically equivalent to the quadratic equation

$$k_2^2 - 2Hk_2 + K = 0.$$

Employing the quadratic formula, we get

$$\begin{aligned} k_2 &= \frac{-(-2H) \pm \sqrt{(-2H)^2 - 4(1)(K)}}{2(1)} \\ &= \frac{2H \pm \sqrt{4(H^2 - K)}}{2} \\ &= H \pm \sqrt{H^2 - K}. \end{aligned}$$

This also means

$$\begin{aligned} k_1 &= 2H - k_2 \\ &= 2H - (H \pm \sqrt{H^2 - K}) \\ &= H \mp \sqrt{H^2 - K}. \end{aligned}$$

As we conventionally require $k_1 > k_2$, we will choose $k_1 = H + \sqrt{H^2 - K}$ and $k_2 = H - \sqrt{H^2 - K}$. Substituting our expressions for H, K, our principal curvatures are

$$\begin{aligned} k_1 &= H + \sqrt{H^2 - K} \\ &= 0 + \sqrt{0^2 - \left(-\frac{c^2}{(v^2 + c^2)^2}\right)^2} \\ &= \frac{c}{v^2 + c^2} \end{aligned}$$

and

$$\begin{aligned} k_2 &= H - \sqrt{H^2 - K} \\ &= 0 - \sqrt{0^2 - \left(-\frac{c^2}{(v^2 + c^2)^2}\right)^2} \\ &= -\frac{c}{v^2 + c^2}, \end{aligned}$$

which are in terms of the variables c, v. ◻

3-5.4. Hyperboloid $z = xy$

Proof. The surface $z = xy$ can be viewed as the surface $z = axy$ in Exercise 3-5.1 with $a = 1$. This means we can just take our results for Exercise 3-5.1 and substitute $a = 1$. The coefficients of our first fundamental form are

$$\begin{aligned} E(u, v) &= 1 + a^2 v^2 \\ &= 1 + (1)^2 v^2 \\ &= 1 + v^2, \end{aligned}$$

as well as

$$\begin{aligned} F(u, v) &= a^2 uv \\ &= (1)^2 uv \\ &= uv \end{aligned}$$

and

$$\begin{aligned} G(u, v) &= 1 + a^2 u^2 \\ &= 1 + (1)^2 u^2 \\ &= 1 + u^2. \end{aligned}$$

The coefficients of our second fundamental form are

$$e = 0,$$

as well as

$$\begin{aligned} f(u, v) &= \frac{a}{\sqrt{a^2(u^2 + v^2) + 1}} \\ &= \frac{(1)}{\sqrt{(1)^2(u^2 + v^2) + 1}} \\ &= \frac{1}{u^2 + v^2 + 1} \end{aligned}$$

and

$$g = 0.$$

The Gaussian curvature is

$$\begin{aligned} K(u,v) &= -\frac{a^2}{(1+a^2(u^2+v^2))^2} \\ &= -\frac{(1)^2}{(1+(1)^2(u^2+v^2))^2} \\ &= -\frac{1}{(1+u^2+v^2)^2}, \end{aligned}$$

and the mean curvature is

$$\begin{aligned} H(u,v) &= -\frac{a^3uv}{(1+a^2(u^2+v^2))^{\frac{3}{2}}} \\ &= -\frac{(1)^3uv}{(1+(1)^2(u^2+v^2))^{\frac{3}{2}}} \\ &= -\frac{uv}{(1+u^2+v^2)^{\frac{3}{2}}}. \end{aligned}$$

Finally, the principal curvatures are

$$\begin{aligned} k_1 &= -\frac{a^3uv}{(1+a^2(u^2+v^2))^{\frac{3}{2}}} + \sqrt{\frac{a^6u^2v^2}{(1+a^2(u^2+v^2))^3} + \frac{a^2}{(1+a^2(u^2+v^2))^2}} \\ &= -\frac{(1)^3uv}{(1+(1)^2(u^2+v^2))^{\frac{3}{2}}} + \sqrt{\frac{(1)^6u^2v^2}{(1+(1)^2(u^2+v^2))^3} + \frac{(1)^2}{(1+(1)^2(u^2+v^2))^2}} \\ &= -\frac{uv}{(1+u^2+v^2)^{\frac{3}{2}}} + \sqrt{\frac{u^2v^2}{(1+u^2+v^2)^3} + \frac{1}{(1+u^2+v^2)^2}} \end{aligned}$$

and

$$\begin{aligned} k_2 &= -\frac{a^3uv}{(1+a^2(u^2+v^2))^{\frac{3}{2}}} - \sqrt{\frac{a^6u^2v^2}{(1+a^2(u^2+v^2))^3} + \frac{a^2}{(1+a^2(u^2+v^2))^2}} \\ &= -\frac{(1)^3uv}{(1+(1)^2(u^2+v^2))^{\frac{3}{2}}} - \sqrt{\frac{(1)^6u^2v^2}{(1+(1)^2(u^2+v^2))^3} + \frac{(1)^2}{(1+(1)^2(u^2+v^2))^2}} \\ &= -\frac{uv}{(1+u^2+v^2)^{\frac{3}{2}}} - \sqrt{\frac{u^2v^2}{(1+u^2+v^2)^3} + \frac{1}{(1+u^2+v^2)^2}}, \end{aligned}$$

as desired. □