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Midterm solutions

1. Determine the curvature and torsion of the helix $\alpha(s) = (3\cos(\frac{s}{5}), 3\sin(\frac{s}{5}), 4(\frac{s}{5}))$ for all $s \in \mathbb{R}$.

Proof. Given $\alpha(s) = (3\cos(\frac{s}{5}), 3\sin(\frac{s}{5}), 4\frac{s}{5})$, we obtain its first derivative

$$\begin{aligned} \alpha'(s) &= \frac{d\alpha}{ds} = \frac{d}{ds}\left(3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), 4\frac{s}{5}\right) \\ &= \left(\frac{d}{ds}\left(3\cos\left(\frac{s}{5}\right)\right), \frac{d}{ds}\left(3\sin\left(\frac{s}{5}\right)\right), \frac{d}{ds}\left(4\frac{s}{5}\right)\right) \\ &= \left(-\frac{3}{5}\sin\left(\frac{s}{5}\right), \frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5}\right) \end{aligned}$$

and its second derivative

$$\begin{aligned} \alpha''(s) &= \frac{d\alpha'}{ds} = \frac{d}{ds}\left(-\frac{3}{5}\sin\left(\frac{s}{5}\right), \frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5}\right) \\ &= \left(\frac{d}{ds}\left(-\frac{3}{5}\sin\left(\frac{s}{5}\right)\right), \frac{d}{ds}\left(\frac{3}{5}\cos\left(\frac{s}{5}\right)\right), \frac{d}{ds}\left(\frac{4}{5}\right)\right) \\ &= \left(-\frac{3}{25}\cos\left(\frac{s}{5}\right), -\frac{3}{25}\sin\left(\frac{s}{5}\right), 0\right). \end{aligned}$$

Since $s \in \mathbb{R}$ is the arc length parameter, according to the definition on page 16 of do Carmo, we obtain the curvature

$$\begin{aligned} k(s) &= |\alpha''(s)| \\ &= \sqrt{\left(-\frac{3}{25}\cos\left(\frac{s}{5}\right)\right)^2 + \left(-\frac{3}{25}\sin\left(\frac{s}{5}\right)\right)^2 + (0)^2} \\ &= \sqrt{\left(\frac{3}{25}\right)^2\left(\cos^2\left(\frac{s}{5}\right) + \sin^2\left(\frac{s}{5}\right)\right)} \\ &= \frac{3}{25}. \end{aligned}$$

Now, we will find the torsion. Recall from page 17 of do Carmo that, since $k(s) = \frac{3}{25} \neq 0$, we have $\alpha''(s) = k(s)n(s)$, which implies $n(s) = \frac{\alpha''(s)}{k(s)}$. (Alternatively, one can find this from one of the Frenet formulas $t' = kn$ and identify $t(s) = \alpha'(s)$ to get $\alpha''(s) = k(s)n(s)$.) So we obtain the normal vector

$$\begin{aligned} n(s) &= \frac{\alpha''(s)}{k(s)} = \frac{\alpha''(s)}{\frac{3}{25}} = \frac{25}{3}\alpha''(s) \\ &= \frac{25}{3}\left(-\frac{3}{25}\cos\left(\frac{s}{5}\right), -\frac{3}{25}\sin\left(\frac{s}{5}\right), 0\right) \\ &= \left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right). \end{aligned}$$

Using this normal vector, we get the binormal vector

$$\begin{aligned}
b(s) &= t(s) \times n(s) \\
&= \alpha'(s) \times n(s) \\
&= \left(-\frac{3}{5}\sin\left(\frac{s}{5}\right), \frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5}\right) \times \left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{3}{5}\sin\left(\frac{s}{5}\right) & \frac{3}{5}\cos\left(\frac{s}{5}\right) & \frac{4}{5} \\ -\cos\left(\frac{s}{5}\right) & -\sin\left(\frac{s}{5}\right) & 0 \end{vmatrix} \\
&= \begin{vmatrix} \frac{3}{5}\cos(\frac{s}{5}) & \frac{4}{5} \\ -\sin(\frac{s}{5}) & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -\frac{3}{5}\sin(\frac{s}{5}) & \frac{4}{5} \\ -\cos(\frac{s}{5}) & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -\frac{3}{5}\sin(\frac{s}{5}) & \frac{3}{5}\cos(\frac{s}{5}) \\ -\cos(\frac{s}{5}) & -\sin(\frac{s}{5}) \end{vmatrix}\mathbf{k} \\
&= \left(\left(\frac{3}{5}\cos\left(\frac{s}{5}\right)\right)(0) - \left(-\sin\left(\frac{s}{5}\right)\right)\left(\frac{4}{5}\right)\right)\mathbf{i} - \left(\left(-\frac{3}{5}\sin\left(\frac{s}{5}\right)\right)(0) - \left(-\cos\left(\frac{s}{5}\right)\right)\left(\frac{4}{5}\right)\right)\mathbf{j} \\
&\quad + \left(\left(-\frac{3}{5}\sin\left(\frac{s}{5}\right)\right)\left(-\sin\left(\frac{s}{5}\right)\right) - \left(-\cos\left(\frac{s}{5}\right)\right)\left(\frac{3}{5}\cos\left(\frac{s}{5}\right)\right)\right)\mathbf{k} \\
&= \frac{4}{5}\sin\left(\frac{s}{5}\right)\mathbf{i} - \frac{4}{5}\cos\left(\frac{s}{5}\right)\mathbf{j} + \frac{3}{5}\left(\sin^2\left(\frac{s}{5}\right) + \cos^2\left(\frac{s}{5}\right)\right)\mathbf{k} \\
&= \frac{4}{5}\sin\left(\frac{s}{5}\right)\mathbf{i} - \frac{4}{5}\cos\left(\frac{s}{5}\right)\mathbf{j} + \frac{3}{5}\mathbf{k} \\
&= \left(\frac{4}{5}\sin\left(\frac{s}{5}\right), -\frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\right),
\end{aligned}$$

along with its derivative

$$\begin{aligned}
b'(s) = \frac{db}{ds} &= \frac{d}{ds}\left(\frac{4}{5}\sin\left(\frac{s}{5}\right), -\frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\right) \\
&= \frac{d}{ds}\left(\frac{4}{5}\sin\left(\frac{s}{5}\right), -\frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\right) \\
&= \left(\frac{d}{ds}\left(\frac{4}{5}\sin\left(\frac{s}{5}\right)\right), \frac{d}{ds}\left(-\frac{4}{5}\cos\left(\frac{s}{5}\right)\right), \frac{d}{ds}\left(\frac{3}{5}\right)\right) \\
&= \left(\frac{4}{25}\cos\left(\frac{s}{5}\right), -\frac{4}{25}\sin\left(\frac{s}{5}\right), 0\right).
\end{aligned}$$

So, from $b'(s) = \tau(s)n(s)$ (c.f. do Carmo, page 18), we obtain the torsion

$$\begin{aligned}
\tau(s) &= b'(s) \cdot n(s) \\
&= \left(\frac{4}{25}\cos\left(\frac{s}{5}\right), -\frac{4}{25}\sin\left(\frac{s}{5}\right), 0\right) \cdot \left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right) \\
&= \left(\frac{4}{25}\cos\left(\frac{s}{5}\right)\right)\left(-\cos\left(\frac{s}{5}\right)\right) + \left(-\frac{4}{25}\sin\left(\frac{s}{5}\right)\right)\left(-\sin\left(\frac{s}{5}\right)\right) + (0)(0) \\
&= -\frac{4}{25}\left(\cos^2\left(\frac{s}{5}\right) + \sin^2\left(\frac{s}{5}\right)\right) \\
&= -\frac{4}{25},
\end{aligned}$$

as desired. □

2. Let $\alpha(s)$ be a unit speed curve lying on the plane with curvature $k(s) = c$, where $c > 0$ is a constant. Prove that $\alpha(s)$ is part of a circle of radius $\frac{1}{c}$.

Proof. Let $n(s)$ be a unit normal vector to $\alpha(s)$ and define

$$\beta(s) := \alpha(s) + \frac{1}{k(s)}n(s).$$

Since $k(s) = c$, we really have

$$\beta(s) = \alpha(s) + \frac{1}{c}n(s)$$

Now, since $\alpha(s)$ is a plane curve, we have zero torsion (i.e. $\tau = 0$), and so one of the Frenet formulas gives us

$$\begin{aligned}
n'(s) &= -k(s)t(s) - \tau(s)b(s) \\
&= -ct(s) - (0)b(s) \\
&= -ct(s) \\
&= -c\alpha'(s).
\end{aligned}$$

So we get the derivative

$$\begin{aligned}\beta'(s) = \frac{d\beta}{ds} &= \frac{d}{ds}\left(\alpha(s) + \frac{1}{c}n(s)\right)\\ &= \alpha'(s) + \frac{1}{c}n'(s)\\ &= \alpha'(s) + \frac{1}{c}(-c\alpha'(s))\\ &= \alpha'(s) - \alpha'(s)\\ &= 0.\end{aligned}$$

Therefore, $\beta(s)$ is a constant vector; in other words, $\beta(s) = p$ for some fixed point $p \in \mathbb{R}^2$. Hence, we get

$$p = \beta(s) = \alpha(s) + \frac{1}{c}n(s),$$

or

$$\alpha(s) - p = -\frac{1}{c}n(s),$$

as well as its associated magnitude

$$\begin{aligned}|\alpha(s) - p| &= \left|-\frac{1}{c}n(s)\right|\\ &= \frac{1}{c}|n(s)|\\ &= \frac{1}{c}.\end{aligned}$$

This establishes that $\alpha(s)$ is a circle of radius $\frac{1}{c}$ that is centered at p. □

3. Determine the unit normal and the equation of the tangent plane of the surface $2x^2 + y^2 + z = 1$ at the point $p = (1, 1, -2)$.

Proof. Define $f(x, y, z) := 2x^2 + y^2 + z$. Then the partial derivatives are $f_x(x, y, z) = 4x$, $f_y(x, y, z) = 2y$, $f_z(x, y, z) = 1$. In particular, at the point $(1, 1, -2)$, we obtain the gradient

$$\begin{aligned}\nabla f(1, 1, -2) &= (f_x(1, 1, -2), f_y(1, 1, -2), f_z(1, 1, -2))\\ &= (4(1), 2(1), 1)\\ &= (4, 2, 1)\end{aligned}$$

and its associated magnitude

$$\begin{aligned}|\nabla f(1, 1, -2)| &= \sqrt{(4)^2 + (2)^2 + (1)^2}\\ &= \sqrt{16 + 4 + 1}\\ &= \sqrt{21}.\end{aligned}$$

So the unit normal at $(1, 1, -2)$ is

$$\begin{aligned}N(1, 1, -2) &= \frac{\nabla f(1, 1, -2)}{|\nabla f(1, 1, -2)|}\\ &= \frac{(4, 2, 1)}{\sqrt{21}}\\ &= \left(\frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}\right).\end{aligned}$$

Now, we recall that the equation of the tangent plane at some point $(x_0, y_0, z_0) \in \mathbb{R}^3$ is given by

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Applying the point $p = (1, 1, -2)$ and our partial derivatives at that point, we get

$$4(x - 1) + 2(y - 1) + 1(z + 2) = 0,$$

or equivalently

$$4x + 2y + z = 4,$$

which is the equation of the tangent plane of $2x^2 + y^2 + z = 1$ at $(1, 1, -2)$. □