Final
Math 146C Section 10

Spring 2020

Please write the pledge below in your own handwriting, fill in the required information, and sign. "On my honor, I (your name) have neither given nor received any unauthorized aid on this examination. I have never used unauthorized tools or look up unauthorized materials. I finish my exam within 13 hours"

- Pledge:

- First Name:
- Last Name:
- Student ID:
- Sign:

Instructions:

- Write above pledge and sign, and without this pledge you will get a 0 for final exam.
- **Show ALL your work to receive credit!** Unless otherwise specified, an answer without explanation might receive no credit. But the formulas or general solutions in lecture notes can be used directly in your answers.
- This is open book. You can use lecture notes, your own notes, or books. But you can not use calculators, and can not search answers online or receive help from others.
- You have 13 hours to complete this exam and to submit to crowdmark.

Question:	1	2	3	4	5	6	7	8	Total
Points:	6	7	6	6	6	7	6	6	50

Question 1 (6 points) Solve the problem

$$u_x + 2u_y = u^2, \quad u(x, 3x) = 1$$

Question 2 (7 points) Solve the problem

$$\begin{aligned} &u_{tt} - u_{xx} = \cos(t) && 0 < x < 1, t > 0, \\ &u_x(0,t) = u_x(1,t) = 0, && t \geq 0, \\ &u(x,0) = 1 + \cos(\pi x), && 0 \leq x \leq 1, \\ &u_t(x,0) = \cos(\pi x)\cos(2\pi x), && 0 \leq x \leq 1 \end{aligned}$$

Please use the eigenfunctions $X_n = \cos(n\pi x)$, $(n = 0, 1, 2, ...)$ in your expansion; you do not need to derive these X_n by the cases 1,2,3.

Question 3 (6 points) The harmonic function $u(x, y)$ satisfies

$$\begin{aligned} \Delta u &= 0, \qquad \text{in} \quad 0 < r < 1, \\ u(x, y) &= x^3, \quad \text{on} \quad r = 1. \end{aligned}$$

in the unit disk $D : r^2 = x^2 + y^2 < 1$. Prove by maximum principle that

$$\max_D |u| \leq 1.$$

Question 4 (6 points) Solve the problem for $w(r, \theta)$ in the disk with radius 2,

$$\begin{aligned} \Delta w &= 0, \qquad \text{in} \quad 0 < r < 2, \\ w(2, \theta) &= x + y^2, \quad \text{on} \quad r = 2. \end{aligned}$$

Find the expression of $w(r, \theta)$ and then express it in terms of x, y. You may directly use the general solution in lecture notes.

Question 5 (6 points) Show that the function $G(x, \xi) = \frac{1}{2}|x - \xi|$ is a fundamental solution of the equation

$$u''(x) = 0, \quad -\infty < x < \infty.$$

By definition of fundamental solution, this is to show that $G(x, \xi)$ satisfies

$$G''(x, \xi) = \delta(x - \xi), \quad -\infty < x < \infty,$$

where prime is the derivative with respect to x.

Question 6 (7 points)

(a). (3 points) Verify that the function $G(x, y, \xi, \eta) = \Gamma(x - \xi, y - \eta) - \Gamma(x + \xi, y - \eta)$ is the Green's function for Laplace equation in the half plane $D : x > 0, -\infty < y < \infty$. Note that $\Gamma(x, y) = -\frac{1}{4\pi} \ln(x^2 + y^2)$. That is to verify that

$$\begin{aligned} &\Delta G = -\delta(x - \xi, y - \eta), \quad \text{in} \quad D \\ &G = 0, \quad \text{on} \quad x = 0, \end{aligned}$$

with parameters $(\xi, \eta) \in D$. You can directly use properties of $\Gamma(x, y)$ in lecture notes.

(b). (2 points) Derive the Poisson's kernel $K = -\partial_n G$ on the boundary $\partial D : x = 0$.

(c). (2 points) With results of (a) and (b), state the general solution (integral representation) for the problem (Just write down the final answer, no need to show steps)

$$\begin{aligned} &\Delta u = 0, \quad \text{in} \quad D \\ &u(0, y) = h(y), \quad \text{on} \quad x = 0. \end{aligned}$$

Question 7 (6 points) For the first order differential equation

$$\begin{aligned} xu_x + uu_y + zu_z &= x \\ u(x, y, 1) &= x + y \end{aligned}$$

(a). (3 points) Write down the characteristic equations (CE) and initial conditions (IC) with independent variables (t, s_1, s_2), by method of characteristics. Do not solve them.

(b). (3 points) Find the Jacobian on initial surface Γ, i.e., at $t = 0$. Is there a unique solution?

Question 8 (6 points) For the wave equation in $\mathbb{R}^3$ (i.e., $(x_1, x_2, x_3) \in \mathbb{R}^3$), let $u = u(r, t)$ be the radial symmetric solution of the system

$$
\begin{aligned}
&u_{tt} = \Delta u, \quad 0 \le r < \infty, \quad t > 0 \\
&u(r, 0) = 1, \quad r \ge 0 \\
&u_t(r, 0) = \begin{cases} 2, & 0 \le r \le 1, \\ 0, & r > 1 \end{cases}
\end{aligned}
$$

Find $u(2, 1)$ and $u(2, 2)$.