Solution to final.

1. $\boxed{6\text{pts}}$

(CE) $\begin{cases} x_t = 1 \\ y_t = 2 \\ u_t = u^2 \end{cases}$, $\to \boxed{1pt}$ (IC) $\begin{cases} x(0,s) = s \\ y(0,s) = 3s \\ u(0,s) = 1 \end{cases} \to \boxed{1pt}$

$$\left.\begin{aligned} x(t,s) &= \int_0^t 1\, dt + x(0,s) = t + s \\ y(t,s) &= \int_0^t 2\, dt + y(0,s) = 2t + 3s \end{aligned}\right\} \to \boxed{1pt}$$

$$\frac{du}{dt} = u^2 \Rightarrow \frac{1}{u^2} du = dt$$

$$-\frac{1}{u} = t + C(s) \Rightarrow u = \frac{-1}{t+C} \to \boxed{1pt}$$

$$u(0,s) = \frac{-1}{C} = 1 \Rightarrow C = -1$$

So $u = \dfrac{1}{1-t} \to \boxed{1pt}$

$$3x - y = t \Rightarrow u = \frac{1}{1+y-3x} \to \boxed{1pt}$$

2. $\boxed{7\text{pts}}$

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos(n\pi x) \longrightarrow \boxed{1\text{pt}}$$

$$\sum_{n=0}^{\infty} T_n'' \cos(n\pi x) + n^2\pi^2 T_n \cos(n\pi x)$$

$$= \cos t \longrightarrow \boxed{1\text{pt}}$$

$$\begin{cases} T_0'' = \cos t \\ T_n'' + n^2\pi^2 T_n = 0, \quad n = 1, 2, \cdots \end{cases} \longrightarrow \boxed{1\text{pt}}$$

$$T_0 = -\cos(t) + \frac{A_0 + B_0 t}{2}$$

$$T_n = A_n \cos(n\pi t) + B_n \sin(n\pi t). \longrightarrow \boxed{1\text{pt}}$$

$$u(x,t) = -\cos t + \frac{A_0 + B_0 t}{2}$$

$$+ \sum_{n=1}^{\infty} \left[A_n \cos(n\pi t) + B_n \sin(n\pi t)\right] \cos(n\pi x)$$

$u(x,0)$

$= -1 + \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = 1 + \cos(\pi x)$

$\Rightarrow \quad A_0 = 4, \quad A_1 = 1.$

$A_n = 0, \quad$ if $n \neq 0, 1. \quad \longrightarrow \boxed{1pt}$

$u_t(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} n\pi B_n \cos(n\pi x)$

$= \cos(\pi x)\cos(2\pi x)$

$= \frac{1}{2}\cos(\pi x) + \frac{1}{2}\cos(3\pi x)$

$B_1 = \frac{1}{2}, \quad B_3 = \frac{1}{2}$

$B_n = 0$ if $n \neq 1, 3. \quad \longrightarrow \boxed{1pt}$

So. $u(x,t) = 2 - \cos t + \cos(\pi t)\cos(\pi x)$

$+ \frac{1}{2}\sin(\pi t)\cos(\pi x)$

$+ \frac{1}{2}\sin(3\pi t)\cos(3\pi x) \quad \longrightarrow \boxed{1pt}$

3. $\boxed{6\text{pts}}$

By MP

$$\max_{D} u \le \max_{\partial D} u \quad \to \boxed{1\text{pt}}$$

$$= \max_{\partial D} x^3$$

$$= \max_{\partial D} (r\cos\theta)^3$$

$$\boxed{r=1} \quad \overset{\downarrow}{\le} 1 \quad \to \boxed{1\text{pt}}$$

$$\min_{D} u \ge \min_{\partial D} u \quad \to \boxed{1\text{pt}}$$

$$= \min_{\partial D} x^3 \quad \to \boxed{1\text{pt}}$$

$$= -1$$

So $-1 \le u \le 1$ in D $\to \boxed{1\text{pt}}$

$|u| \le 1$ in D

$$\Rightarrow \max_{D} |u| \le 1. \quad \to \boxed{1\text{pt}}$$

4. $\boxed{6\text{pts}}$

$$w(r,\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n \left[\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)\right] \rightarrow \boxed{1\text{pt}}$$

On $r=2$:

$$h(\theta) = x + y^2$$
$$= r\cos\theta + (r\sin\theta)^2$$
$$= 2\cos\theta + 4\sin^2\theta \longrightarrow \boxed{1\text{pt}}$$

$$w(2,\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} 2^n \left[\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)\right]$$
$$= 2\cos\theta + 4\sin^2\theta$$
$$= 2\cos\theta + 2(1-\cos 2\theta) \longrightarrow \boxed{1\text{pt}}$$

$$\Rightarrow \begin{cases} \frac{\alpha_0}{2} = 2 & \Rightarrow \quad \alpha_0 = 4 \\ 2\alpha_1 = 2 & \Rightarrow \quad \alpha_1 = 1 \\ 4\alpha_2 = -2 & \Rightarrow \quad \alpha_2 = -\frac{1}{2} \end{cases}$$

$\alpha_n = 0, \quad$ if $n \geq 3.$ $\rightarrow$ $\boxed{1pt}$

$\beta_n = 0 \quad$ for all $n \geq 1$

$\Rightarrow \quad w(r,\theta) = 2 + r\cos\theta - \frac{1}{2} r^2 \cos(2\theta)$

$\rightarrow$ $\boxed{1pt}$

$w(x,y) = 2 + x - \frac{1}{2} r^2 [\cos^2\theta - \sin^2\theta]$

$= 2 + x - \frac{1}{2}(x^2 - y^2)$

$\rightarrow$ $\boxed{1pt}$

5. $\boxed{6\text{pts}}$

$$G(x,\xi) = \begin{cases} \frac{1}{2}(x-\xi) & \text{if } x > \xi \\ \frac{1}{2}(\xi - x) & \text{if } x < \xi. \end{cases} \longrightarrow \boxed{1\text{pt}}$$

$$G'(x,\xi) = \begin{cases} \frac{1}{2}, & \text{if } x > \xi \\ -\frac{1}{2}, & \text{if } x < \xi \end{cases} \longrightarrow \boxed{1\text{pt}}$$

$$G''(x) = \begin{cases} 0, & \text{if } x > \xi \\ 0, & \text{if } x < \xi \end{cases} \longrightarrow \boxed{1\text{pt}}$$

For any smooth $u(x)$

$$\int_{-\infty}^{\infty} u(x)\, G''(x,\xi)\, dx$$

$$= \int_{\xi-\varepsilon}^{\xi+\varepsilon} u(x)\, G''(x,\xi)\, dx$$

$$= \underbrace{u\, G' \Big|_{\xi-\varepsilon}^{\xi+\varepsilon}}_{(i)} - \underbrace{\int_{\xi-\varepsilon}^{\xi+\varepsilon} u' G'\, dx}_{(ii)} \longrightarrow \boxed{1\text{pt}}$$

For (i)

$$u G' \Big|_{\xi-\varepsilon}^{\xi+\varepsilon} = u(\xi+\varepsilon)\cdot\frac{1}{2} + u(\xi-\varepsilon)\cdot\frac{1}{2}$$

$$\to u(\xi) \quad \text{as } \varepsilon \to 0.$$

1pt

For (ii)

$$|u| < C,\ |G| < C,\ |u'| < C,\ |G'| < C$$

$$\Rightarrow \int_{\xi-\varepsilon}^{\xi+\varepsilon} G' u' \, dx \to 0 \quad \text{as } \varepsilon \to 0.$$

So $\int_{-\infty}^{\infty} u(x)\, G''(x,\xi)\, dx = u(\xi)$

$$\Rightarrow \quad G''(x,\xi) = \delta(x-\xi).$$

1pt

6. $\boxed{7\text{pts}}$

(a) $\Delta G = \Delta\Gamma(x-\xi, y-\eta)$
$- \Delta\Gamma(x+\xi, y-\eta)$ in D
$= -\delta(x-\xi, y-\eta) - 0$
$= -\delta(x-\xi, y-\eta) \rightarrow \boxed{1\text{pt}}$

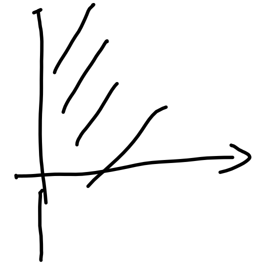
at $x=0$.

$$G = -\frac{1}{4\pi}\ln\left[(x-\xi)^2+(y-\eta)^2\right] + \frac{1}{4\pi}\ln\left[(x+\xi)^2+(y-\eta)^2\right] \rightarrow \boxed{1\text{pt}}$$

$$= -\frac{1}{4\pi}\ln\left[\xi^2+(y-\eta)^2\right] + \frac{1}{4\pi}\ln\left[\xi^2+(y-\eta)^2\right]$$

$$= 0. \rightarrow \boxed{1\text{pt}}$$

(b) At $x=0$



$$K = -\partial_n G = \partial_x G$$

$$= -\frac{1}{4\pi}\frac{2(x-\xi)}{(x-\xi)^2+(y-\eta)^2} + \frac{1}{4\pi}\frac{2(x+\xi)}{\left[(x+\xi)^2+(y-\eta)^2\right]}$$

$\boxed{\text{1pt}}$

$$\overset{x=0}{=} -\frac{1}{4\pi}\frac{-2\xi}{\xi^2+(y-\eta)^2} + \frac{1}{4\pi}\cdot\frac{2\xi}{\xi^2+(y-\eta)^2}$$

$$= \frac{1}{\pi}\frac{\xi}{\xi^2+(y-\eta)^2}$$

$\boxed{\text{1pt}}$

(c)

$$u(\xi,\eta) = \int_{-\infty}^{\infty}\frac{1}{\pi}\frac{\xi}{\xi^2+(y-\eta)^2}h(y)\,dy.$$

$\boxed{\text{1pt}}$

$\boxed{\text{1pt}}$

7. $\boxed{6\text{pts}}$

(a)

CE: $\begin{cases} x_t = x & (1) \\ y_t = u & (2) \\ z_t = z & (3) \\ u_t = x & (4) \end{cases}$

(IC) $\begin{cases} x(0, s_1, s_2) = s_1 & (5) \\ y(0, s_1, s_2) = s_2 & (6) \\ z(0, s_1, s_2) = 1 & (7) \\ u(0, s_1, s_2) = s_1 + s_2 & (8) \end{cases}$

correct any 3 Equations $\rightarrow$ 1pt

correct any 4-6 Equations $\rightarrow$ 2pts

correct all Equations $\rightarrow$ 3pts

(b)

$$J\Big|_{T} = \begin{vmatrix} x_{s_1} & y_{s_1} & z_{s_1} \\ x_{s_2} & y_{s_2} & z_{s_2} \\ x_t & y_t & z_t \end{vmatrix}\Bigg|_{t=0} \longrightarrow \boxed{1pt}$$

$$= \begin{vmatrix} (s_1)_{s_1} & (s_2)_{s_1} & (1)_{s_1} \\ (s_1)_{s_2} & (s_2)_{s_2} & (1)_{s_2} \\ x & y & z \end{vmatrix}\Bigg|_{t=0}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_1 & s_1+s_2 & 1 \end{vmatrix} = 1 \longrightarrow \boxed{1pt}$$

Yes. there's a unique solution $\longrightarrow \boxed{1pt}$

8. $\boxed{6\text{pts}}$

$f(r) = 1, \quad \tilde{f}(r) = 1. \longrightarrow \boxed{1\text{pt}}$

$\tilde{g}(r) = \begin{cases} 2, & |r| \le 1 \\ 0, & |r| > 1. \end{cases} \longrightarrow \boxed{1\text{pt}}$

$c = 1$

$u(r,t) = \frac{1}{2r}[(r+t) + (r-t)]$

$+ \frac{1}{2r}\int_{r-t}^{r+t} s\,\tilde{g}(s)\,ds \longrightarrow \boxed{1\text{pt}}$

$= \frac{1}{2} + \frac{1}{2r}\int_{r-t}^{r+t} s\,\tilde{g}(s)\,ds \longrightarrow \boxed{1\text{pt}}$

So: $u(2,1) = \frac{1}{2} + \frac{1}{4}\int_{1}^{3} s\,\tilde{g}\,ds$

$= \frac{1}{2} + 0 = \frac{1}{2} \longrightarrow \boxed{1\text{pt}}$

$u(2,2) = \frac{1}{2} + \frac{1}{4}\int_{0}^{4} s\,\tilde{g}\,ds.$

$= \frac{1}{2} + \frac{1}{4}\int_{0}^{1} s \cdot 2\,ds \longrightarrow \boxed{1\text{pt}}$

$= \frac{1}{2} + \frac{1}{4} s^2\Big|_0^1 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$