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Homework 11 solutions

1. Solve the problem

$$\begin{aligned} \Delta u &= 0 && 0 < x < b, 0 < y < d, \\ u(0, y) = u(b, y) &= 0 && 0 < y < d, \\ u(x, 0) = h(x), u(x, d) &= k(x) && 0 < x < b. \end{aligned}$$

Derive the general solution as done in Lecture 12 (May 13), including the formulas for the Fourier coefficients appearing in the general solution.

Remark. *On your homework and exam, please disregard my derivation of the general solution using separation of variables. I only derive the general solution here in order to give a complete picture. If you see a problem like this on your final exam, with vertical boundary conditions being zero), please start your own exam solution with your choice of either of the following expressions of the general solution:*

$$u(x, y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{b}x\right)(A_n e^{\frac{n\pi}{b}y} + B_n e^{-\frac{n\pi}{b}y}), \tag{1}$$

$$u(x, y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{b}x\right)\left(A_n \sinh\left(\frac{n\pi}{b}y\right) + B_n \sinh\left(\frac{n\pi}{b}(y - d)\right)\right), \tag{2}$$

Expression (2) can be found at the bottom of page 189 of the textbook and is highly recommended because it requires fewer algebraic manipulations than expression (1). In my solution below, I will attempt to derive both expressions. But also observe that the list of functions $\{\sinh(\frac{n\pi}{b}y), \sinh(\frac{n\pi}{b}(y-d)\}$ is a linear combination of the list of functions $\{e^{\frac{n\pi}{b}y}, e^{-\frac{n\pi}{b}y}\}$, and vice versa. If you are given a problem with horizontal boundary conditions being zero, do not use these formulas; see instead the general solution in the professor's Lecture 12 notes.

Solution - derive (1) and (2) using the method of separation of variables. We want to find a solution of the form

$$u(x, y) = X(x)Y(y).$$

Our partial derivatives are

$$\begin{aligned} u_{xx}(x, y) &= X_{xx}(x)Y(y), \\ u_{yy}(x, y) &= X(x)Y_{yy}(y) \end{aligned}$$

So the partial differential equation

$$u_{xx} + u_{yy} = \Delta u = 0$$

becomes

$$X_{xx}(x)Y(y) + X(x)Y_{yy}(y) = 0,$$

which we can algebraically rearrange to write

$$\frac{X_{xx}(x)}{X(x)} = -\frac{Y_{yy}(y)}{Y(y)} = -\lambda,$$

where λ is a constant in both x and y. This produces the system of two ordinary differential equations

$$\begin{aligned} \frac{d^2X}{dx^2} + \lambda X &= 0 \\ \frac{d^2Y}{dy^2} - \lambda Y &= 0. \end{aligned}$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$X(x) = \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0, \\ C_1 x + C_2 & \text{if } \lambda = 0, \\ C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) & \text{if } \lambda > 0, \end{cases}$$

$$Y(y) = \begin{cases} D_1 \cos(\sqrt{-\lambda}y) + D_2 \sin(\sqrt{-\lambda}y) & \text{if } \lambda < 0, \\ D_1 y + D_2 & \text{if } \lambda = 0, \\ D_1 e^{\sqrt{\lambda}y} + D_2 e^{-\sqrt{-\lambda}y} & \text{if } \lambda > 0, \end{cases}$$

where C_1, C_2, D_1, D_2 are constants. Now, the boundary conditions

$$u(0, y) = u(b, y) = 0$$

are equivalent to

$$\begin{aligned} X(0)Y(y) &= 0, \\ X(b)Y(y) &= 0, \end{aligned}$$

which imply either $Y(y) = 0$ or $X(0) = X(b) = 0$. If $Y(y) = 0$, then we would have

$$\begin{aligned} u(x, y) &= X(x)Y(y) \\ &= X(x)0 \\ &= 0, \end{aligned}$$

which would be a trivial solution. But we are really interested in finding a nontrivial solution. So we should assume

$$X(0) = X(b) = 0,$$

which will impose constraints on the constants C_1, C_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then

  $$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, \\ X(0) &= 0 \end{aligned}$$

  implies $C_2 = -C_1$, and so we have

  $$\begin{aligned} X(x) &= C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \\ &= C_1 e^{\sqrt{-\lambda}x} - C_1 e^{-\sqrt{-\lambda}x} \\ &= C_1 (e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}). \end{aligned}$$

  Now, if $\lambda < 0$, then $e^{\sqrt{-\lambda}b} - e^{-\sqrt{-\lambda}b} \neq 0$. This means

  $$\begin{aligned} X(x) &= C_1 (e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}), \\ X(b) &= 0 \end{aligned}$$

  implies $C_1 = 0$, and so we have

  $$\begin{aligned} X(x) &= C_1 (e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}) \\ &= 0(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}) \\ &= 0. \end{aligned}$$

  Therefore, we have

  $$\begin{aligned} u(x, y) &= X(x)Y(y) \\ &= 0Y(y) \\ &= 0, \end{aligned}$$

  which is a trivial solution.
- Case 2: Suppose $\lambda = 0$. Then

  $$\begin{aligned} X(x) &= C_1 x + C_2, \\ X(0) &= 0 \end{aligned}$$

  implies $C_2 = 0$, and so we have

  $$\begin{aligned} X(x) &= C_1 x + C_2 \\ &= C_1 x + 0 \\ &= C_1 x. \end{aligned}$$

  Next,

  $$\begin{aligned} X(x) &= C_1 x, \\ X(b) &= 0 \end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 x \\ &= 0x \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} u(x,y) &= X(x)Y(y) \\ &= 0Y(y) \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 3: Suppose $\lambda > 0$. Then

$$\begin{aligned} &X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x), \\ &X(0) = 0 \end{aligned}$$

implies $C_1 = 0$, and so we have

$$\begin{aligned} X(x) &= C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \\ &= 0\cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \\ &= C_2 \sin(\sqrt{\lambda}x). \end{aligned}$$

Next,

$$\begin{aligned} &X(x) = C_2 \sin(\sqrt{\lambda}x), \\ &X(b) = 0 \end{aligned}$$

implies $\sin(\sqrt{\lambda}b) = 0$, which in turn implies $\sqrt{\lambda}b = n\pi$, or equivalently

$$\lambda_n = \lambda = \left(\frac{n\pi}{b}\right)^2,$$

and so we have

$$\begin{aligned} X_n(x) &= C_{2,n} \sin(\sqrt{\lambda_n}x) \\ &= C_{2,n} \sin\left(\frac{n\pi}{b}x\right) \end{aligned}$$

and

$$\begin{aligned} Y_n(y) &= D_{1,n}e^{\sqrt{\lambda_n}y} + D_{2,n}e^{-\sqrt{\lambda_n}y} \\ &= D_{1,n}e^{\sqrt{(\frac{n\pi}{b})^2}y} + D_{2,n}e^{-\sqrt{(\frac{n\pi}{b})^2}y} \\ &= D_{1,n}e^{\frac{n\pi}{b}y} + D_{2,n}e^{-\frac{n\pi}{b}y} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, if we write $A_n = C_{2,n}D_{1,n}$ and $B_n = C_{2,n}D_{2,n}$, then we have

$$\begin{aligned} u_n(x,y) &= X_n(x)Y_n(y) \\ &= \left(C_{2,n} \sin\left(\frac{n\pi}{b}x\right)\right)\left(D_{1,n}e^{\frac{n\pi}{b}y} + D_{2,n}e^{-\frac{n\pi}{b}y}\right) \\ &= \sin\left(\frac{n\pi}{b}x\right)\left(C_{2,n}D_{1,n}e^{\frac{n\pi}{b}y} + C_{2,n}D_{2,n}e^{-\frac{n\pi}{b}y}\right) \\ &= \sin\left(\frac{n\pi}{b}x\right)\left(A_n e^{\frac{n\pi}{b}y} + B_n e^{-\frac{n\pi}{b}y}\right) \end{aligned}$$

for $n = 1, 2, 3, \ldots$, which is a nontrivial solution. If one wishes to take a step further and express the solution in terms of hyperbolic functions, as accomplished by expression (2) in my remark, we can first rewrite

$$\begin{aligned} Y_n(y) &= D_{1,n}e^{\frac{n\pi}{b}y} + D_{2,n}e^{-\frac{n\pi}{b}y} \\ &= 2D_{1,n}\left(\frac{e^{\frac{n\pi}{b}y} - e^{-\frac{n\pi}{b}y}}{2} + \frac{e^{-\frac{n\pi}{b}y}}{2}\right) - 2e^{-\frac{n\pi}{b}d}D_{2,n}\left(\frac{e^{\frac{n\pi}{b}(y-d)} - e^{-\frac{n\pi}{b}(y-d)}}{2} - \frac{e^{\frac{n\pi}{b}(y-d)}}{2}\right) \\ &= 2D_{1,n}\left(\sinh\left(\frac{n\pi}{b}y\right) + \frac{e^{-\frac{n\pi}{b}y}}{2}\right) - 2e^{-\frac{n\pi}{b}d}D_{2,n}\left(\sinh\left(\frac{n\pi}{b}(y-d)\right) - \frac{e^{\frac{n\pi}{b}(y-d)}}{2}\right) \\ &= 2D_{1,n}\sinh\left(\frac{n\pi}{b}y\right) + D_{1,n}e^{-\frac{n\pi}{b}y} - 2e^{-\frac{n\pi}{b}d}D_{2,n}\sinh\left(\frac{n\pi}{b}(y-d)\right) + e^{-2\frac{n\pi}{b}d}D_{2,n}e^{\frac{n\pi}{b}y} \\ &= 2D_{1,n}\sinh\left(\frac{n\pi}{b}y\right) - 2e^{-\frac{n\pi}{b}d}D_{2,n}\sinh\left(\frac{n\pi}{b}(y-d)\right) + e^{-2\frac{n\pi}{b}d}D_{2,n}e^{\frac{n\pi}{b}y} + D_{1,n}e^{-\frac{n\pi}{b}y}. \end{aligned}$$

As the choice of constants is arbitrary, we are allowed to relabel the constants. By relabeling the constants, we can write

$$\begin{aligned}
Y_n(y) &= D_{1,n}\sinh\left(\frac{n\pi}{b}y\right) + D_{2,n}\sinh\left(\frac{n\pi}{b}(y-d)\right) + D_{3,n}e^{\frac{n\pi}{b}y} - D_{3,n}e^{-\frac{n\pi}{b}y} \\
&= D_{1,n}\sinh\left(\frac{n\pi}{b}y\right) + D_{2,n}\sinh\left(\frac{n\pi}{b}(y-d)\right) + 2D_{3,n}\frac{e^{-\frac{n\pi}{b}y} - e^{\frac{n\pi}{b}y}}{2} \\
&= D_{1,n}\sinh\left(\frac{n\pi}{b}y\right) + D_{2,n}\sinh\left(\frac{n\pi}{b}(y-d)\right) + 2D_{3,n}\sinh\left(\frac{n\pi}{b}y\right) \\
&= (D_{1,n} + 2D_{3,n})\sinh\left(\frac{n\pi}{b}y\right) + D_{2,n}\sinh\left(\frac{n\pi}{b}(y-d)\right).
\end{aligned}$$

By relabeling the constants one more time, we can finally write

$$Y_n(y) = D_{1,n}\sinh\left(\frac{n\pi}{b}y\right) + D_{2,n}\sinh\left(\frac{n\pi}{b}(y-d)\right).$$

Therefore, if we write $A_n := C_{2,n}D_{1,n}$ and $B_n := C_{2,n}D_{2,n}$, then we have

$$\begin{aligned}
u_n(x,y) &= X_n(x)Y_n(y) \\
&= \left(C_{2,n}\sin\left(\frac{n\pi}{b}x\right)\right)\left(D_{1,n}\sinh\left(\frac{n\pi}{b}y\right) + D_{2,n}\sinh\left(\frac{n\pi}{b}(y-d)\right)\right) \\
&= \sin\left(\frac{n\pi}{b}x\right)\left(C_{2,n}D_{1,n}\sinh\left(\frac{n\pi}{b}y\right) + C_{2,n}D_{2,n}\sinh\left(\frac{n\pi}{b}(y-d)\right)\right) \\
&= \sin\left(\frac{n\pi}{b}x\right)\left(A_n\sinh\left(\frac{n\pi}{b}y\right) + B_n\sinh\left(\frac{n\pi}{b}(y-d)\right)\right).
\end{aligned}$$

for $n = 1, 2, 3, \ldots$, which is a nontrivial solution.

We recall that an addition of solutions is again a solution. Therefore,

$$u(x,y) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{b}x\right)(A_n e^{\frac{n\pi}{b}y} + B_n e^{-\frac{n\pi}{b}y}), \tag{1}$$

$$u(x,y) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{b}x\right)\left(A_n\sinh\left(\frac{n\pi}{b}y\right) + B_n\sinh\left(\frac{n\pi}{b}(y-d)\right)\right) \tag{2}$$

are general solutions of the problem. □

Solution - compute the Fourier coefficients using hyperbolic sine functions. Given

$$u(x,y) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{b}x\right)\left(A_n\sinh\left(\frac{n\pi}{b}y\right) + B_n\sinh\left(\frac{n\pi}{b}(y-d)\right)\right), \tag{2}$$

we have

$$\begin{aligned}
h(x) = u(x,0) &= \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi}{b}x\right)\sinh\left(-\frac{n\pi}{b}d\right), \\
k(x) = u(x,d) &= \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi}{b}x\right)\sinh\left(\frac{n\pi}{b}d\right).
\end{aligned}$$

Now, recall

$$\int_0^b \sin\left(\frac{n\pi}{b}x\right)\sin\left(\frac{m\pi}{b}x\right)\,dx = \begin{cases}\frac{b}{2} & \text{if } n = m, \\ 0 & \text{if } n \neq m.\end{cases}$$

Consequently, we obtain

$$\begin{aligned}
\alpha_n := \int_0^b h(x)\sin\left(\frac{n\pi}{b}x\right)\,dx &= \int_0^b u(x,0)\sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= \int_0^b \sum_{m=1}^{\infty}\sin\left(\frac{m\pi}{b}x\right)B_m\sin\left(\frac{m\pi}{b}x\right)\sinh\left(-\frac{m\pi}{b}d\right)\sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= \sum_{m=1}^{\infty} B_m\sinh\left(-\frac{m\pi}{b}d\right)\int_0^b \sin\left(\frac{m\pi}{b}x\right)\sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= B_n\sinh\left(-\frac{n\pi}{b}d\right)\frac{b}{2} \\
&= -B_n\sinh\left(\frac{n\pi}{b}d\right)\frac{b}{2}
\end{aligned}$$

and

$$\begin{aligned}
\beta_n := \int_0^b k(x) \sin\left(\frac{n\pi}{b}x\right)\,dx = \int_0^b u(x,d) \sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= \int_0^b \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi}{b}x\right) \sinh\left(\frac{m\pi}{b}d\right) \sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= \sum_{m=1}^{\infty} A_m \sinh\left(\frac{m\pi}{b}d\right) \int_0^b \sin\left(\frac{m\pi}{b}x\right) \sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= A_n \sinh\left(\frac{n\pi}{b}d\right) \frac{b}{2}.
\end{aligned}$$

So we obtain the coefficients

$$\begin{aligned}
A_n &= \frac{2}{b} \frac{1}{\sinh(\frac{n\pi}{b}d)} \beta_n, \\
B_n &= -\frac{2}{b} \frac{1}{\sinh(\frac{n\pi}{b}d)} \alpha_n.
\end{aligned}$$

So our formal solution is

$$\begin{aligned}
u(x,y) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{b}x\right) \left(A_n \sinh\left(\frac{n\pi}{b}y\right) + B_n \sinh\left(\frac{n\pi}{b}(y-d)\right)\right) \\
&= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{b}x\right) \left(\frac{2}{b} \frac{1}{\sinh(\frac{n\pi}{b}d)} \beta_n \sinh\left(\frac{n\pi}{b}y\right) - \frac{2}{b} \frac{1}{\sinh(\frac{n\pi}{b}d)} \alpha_n \sinh\left(\frac{n\pi}{b}(y-d)\right)\right) \\
&= \boxed{\frac{2}{b} \sum_{n=1}^{\infty} \frac{1}{\sinh(\frac{n\pi}{b}d)} \sin\left(\frac{n\pi}{b}x\right) \left(\beta_n \sinh\left(\frac{n\pi}{b}y\right) - \alpha_n \sinh\left(\frac{n\pi}{b}(y-d)\right)\right)},
\end{aligned}$$

where

$$\begin{aligned}
\alpha_n &:= \boxed{\int_0^b h(x) \sin\left(\frac{n\pi}{b}x\right)\,dx}, \\
\beta_n &:= \boxed{\int_0^b k(x) \sin\left(\frac{n\pi}{b}x\right)\,dx},
\end{aligned}$$

as desired. □

Alternate solution - compute the Fourier coefficients using exponential functions. Given

$$u(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{b}x\right) (A_n e^{\frac{n\pi}{b}y} + B_n e^{-\frac{n\pi}{b}y}), \tag{1}$$

we have

$$\begin{aligned}
h(x) = u(x,0) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{b}x\right) (A_n + B_n), \\
k(x) = u(x,d) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{b}x\right) (A_n e^{\frac{n\pi}{b}d} + B_n e^{-\frac{n\pi}{b}d}).
\end{aligned}$$

Now, recall

$$\int_0^b \sin\left(\frac{n\pi}{b}x\right) \sin\left(\frac{m\pi}{b}x\right)\,dx = \begin{cases} \frac{b}{2} & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Consequently, we obtain

$$\begin{aligned}
\alpha_n := \int_0^b h(x) \sin\left(\frac{n\pi}{b}x\right)\,dx = \int_0^b u(x,0) \sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= \int_0^b \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{b}x\right) (A_m + B_m) \sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= \sum_{m=1}^{\infty} (A_m + B_m) \int_0^b \sin\left(\frac{m\pi}{b}x\right) \sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= (A_n + B_n) \frac{b}{2}
\end{aligned}$$

and

$$
\begin{aligned}
\beta_n := \int_0^b k(x)\sin\left(\frac{n\pi}{b}x\right)\,dx = \int_0^b u(x,d)\sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= \int_0^b \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{b}x\right)(A_m e^{\frac{m\pi}{b}d} + B_m e^{-\frac{m\pi}{b}d})\sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= \sum_{m=1}^{\infty}(A_m e^{\frac{m\pi}{b}d} + B_m e^{-\frac{m\pi}{b}d})\int_0^b \sin\left(\frac{m\pi}{b}x\right)\sin\left(\frac{n\pi}{b}x\right)\,dx \\
&= (A_n e^{\frac{n\pi}{b}d} + B_n e^{-\frac{n\pi}{b}d})\frac{b}{2}.
\end{aligned}
$$

In other words, we have the system

$$
\begin{aligned}
A_n + B_n &= \frac{2}{b}\alpha_n, \\
A_n e^{\frac{n\pi}{b}d} + B_n e^{-\frac{n\pi}{b}d} &= \frac{2}{b}\beta_n,
\end{aligned}
$$

which we can solve simultaneously to obtain the coefficients

$$
\begin{aligned}
A_n &= \frac{1}{b}\frac{1}{\sinh(\frac{n\pi}{b}d)}(\beta_n - e^{-\frac{n\pi}{b}d}\alpha_n), \\
B_n &= \frac{1}{b}\frac{1}{\sinh(\frac{n\pi}{b}d)}(e^{\frac{n\pi}{b}d}\alpha_n - \beta_n).
\end{aligned}
$$

So our formal solution is

$$
\begin{aligned}
u(x,y) &= \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{b}x\right)(A_n e^{\frac{n\pi}{b}y} + B_n e^{-\frac{n\pi}{b}y}) \\
&= \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{b}x\right)\left(\frac{1}{b}\frac{1}{\sinh(\frac{n\pi}{b}d)}(\beta_n - e^{-\frac{n\pi}{b}d}\alpha_n)e^{\frac{n\pi}{b}y} + \frac{1}{b}\frac{1}{\sinh(\frac{n\pi}{b}d)}(e^{\frac{n\pi}{b}d}\alpha_n - \beta_n)e^{-\frac{n\pi}{b}y}\right) \\
&= \frac{1}{b}\sum_{n=1}^{\infty}\frac{1}{\sinh(\frac{n\pi}{b}d)}\sin\left(\frac{n\pi}{b}x\right)((\beta_n - e^{-\frac{n\pi}{b}d}\alpha_n)e^{\frac{n\pi}{b}y} + (e^{\frac{n\pi}{b}d}\alpha_n - \beta_n)e^{-\frac{n\pi}{b}y}) \\
&= \frac{1}{b}\sum_{n=1}^{\infty}\frac{1}{\sinh(\frac{n\pi}{b}d)}\sin\left(\frac{n\pi}{b}x\right)(\beta_n(e^{\frac{n\pi}{b}y} - e^{-\frac{n\pi}{b}y}) - \alpha_n(e^{\frac{n\pi}{b}(y-d)} - e^{-\frac{n\pi}{b}(y-d)})) \\
&= \frac{2}{b}\sum_{n=1}^{\infty}\frac{1}{\sinh(\frac{n\pi}{b}d)}\sin\left(\frac{n\pi}{b}x\right)\left(\beta_n\frac{e^{\frac{n\pi}{b}y} - e^{-\frac{n\pi}{b}y}}{2} - \alpha_n\frac{e^{\frac{n\pi}{b}(y-d)} - e^{-\frac{n\pi}{b}(y-d)}}{2}\right) \\
&= \boxed{\frac{2}{b}\sum_{n=1}^{\infty}\frac{1}{\sinh(\frac{n\pi}{b}d)}\sin\left(\frac{n\pi}{b}x\right)\left(\beta_n\sinh\left(\frac{n\pi}{b}y\right) - \alpha_n\sinh\left(\frac{n\pi}{b}(y-d)\right)\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_n &:= \boxed{\int_0^b h(x)\sin\left(\frac{n\pi}{b}x\right)\,dx}, \\
\beta_n &:= \boxed{\int_0^b k(x)\sin\left(\frac{n\pi}{b}x\right)\,dx},
\end{aligned}
$$

as desired. □