Homework 12 solutions

1. Solve the problem for $w(r,\theta)$ on the unit disk

$$\Delta w = 0 \qquad 0 < r < 1, 0 \le \theta \le 2\pi$$
$$w(1,\theta) = x^2 + y \quad 0 \le \theta \le 2\pi.$$

Find the expression for $w(r,\theta)$. Then express it in terms of x and y.

Remark. *I addressed once more in my own solution the three cases of λ again for the purpose of completeness. At this point, it is understood that you have learned from Chapter 5 of the textbook that the cases $\lambda \le 0$ (Dirichlet problems) or $\lambda < 0$ (Neumann problems) yield trivial solutions. According to the professor, you do not need to include in your homework solutions any further work for trivial solutions at this point. This is particularly important to keep in mind when you are submitting the remaining homework or when you are taking your final exam.*

Solution. We want to find a solution of the form

$$w(r,\theta) = R(r)\Theta(\theta).$$

Our partial derivatives are

$$w_r(r,\theta) = R_r(r)\Theta(\theta),$$
$$w_{rr}(r,\theta) = R_{rr}(r)\Theta(\theta),$$
$$w_{\theta\theta}(r,\theta) = R(r)\Theta_{\theta\theta}(\theta).$$

So the partial differential equation

$$w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} = \Delta w = 0$$

becomes

$$R_{rr}(r)\Theta(\theta) + \frac{1}{r}R_r(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta_{\theta\theta}(\theta) = 0,$$

which we can algebraically rearrange to write

$$-\frac{r^2R_{rr}(r) + rR_r(r)}{R(r)} = \frac{\Theta_{\theta\theta}(\theta)}{\Theta(\theta)} = -\lambda,$$

where λ is a constant in both r and θ. This produces the system of two ordinary differential equations

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0$$
$$\frac{d^2\Theta}{dr^2} + \lambda\Theta = 0.$$

This system is decoupled, which allows us to solve each one independently and obtain the general solutions

$$R(r) = \begin{cases} C_1\cos(\sqrt{-\lambda}\ln(r)) + C_2\sin(\sqrt{-\lambda}\ln(r)) & \text{if } \lambda < 0, \\ C_1\ln(r) + C_2 & \text{if } \lambda = 0, \\ C_1r^{\sqrt{\lambda}} + C_2r^{-\sqrt{\lambda}} & \text{if } \lambda > 0, \end{cases}$$

$$\Theta(\theta) = \begin{cases} D_1e^{\sqrt{-\lambda}\theta} + D_2e^{-\sqrt{-\lambda}\theta} & \text{if } \lambda < 0, \\ D_1\theta + D_2 & \text{if } \lambda = 0, \\ D_1\cos(\sqrt{\lambda}\theta) + D_2\sin(\sqrt{\lambda}\theta) & \text{if } \lambda > 0, \end{cases}$$

where C_1, C_2, D_1, D_2 are constants. Now, according to page 196 of the textbook, the equation for Θ holds at the interval $(0, 2\pi)$. In order for $\Theta(\theta)$ to be twice differentiable (so that $\frac{d^2\Theta}{d\theta^2}$ makes sense, after all) for all $\theta \in \mathbb{R}$, we need to impose the periodic boundary conditions

$$\Theta(0) = \Theta(2\pi),$$
$$\frac{d}{d\theta}\Theta(0) = \frac{d}{d\theta}\Theta(2\pi),$$

which will impose constraints on the constants D_1, D_2, depending on λ. This motivates us to break this down into cases.

- Case 1: Suppose $\lambda < 0$. Then we have

$$\Theta(\theta) = D_1 e^{\sqrt{-\lambda}\theta} + D_2 e^{-\sqrt{-\lambda}\theta},$$
$$\Theta(0) = \Theta(2\pi),$$

which implies $D_1 + D_2 = D_1 e^{2\pi\sqrt{-\lambda}} + D_2 e^{-2\pi\sqrt{-\lambda}}$. We also have

$$\frac{d}{d\theta}\Theta(\theta) = \sqrt{-\lambda}(D_1 e^{\sqrt{-\lambda}x} - D_2 e^{-\sqrt{-\lambda}x}),$$
$$\frac{d}{d\theta}\Theta(0) = \frac{d}{d\theta}\Theta(2\pi),$$

which implies $D_1 - D_2 = D_1 e^{2\pi\sqrt{-\lambda}} - D_2 e^{-2\pi\sqrt{-\lambda}}$. Now we will solve for the constants D_1, D_2. We have formulated the linear system

$$D_1 + D_2 = D_1 e^{2\pi\sqrt{-\lambda}} + D_2 e^{-2\pi\sqrt{-\lambda}},$$
$$D_1 - D_2 = D_1 e^{2\pi\sqrt{-\lambda}} - D_2 e^{-2\pi\sqrt{-\lambda}},$$

and we can algebraically rearrange each equation in the system to write

$$D_1(1 - e^{2\pi\sqrt{-\lambda}}) = -D_2(1 - e^{-2\pi\sqrt{-\lambda}}),$$
$$D_1(1 - e^{2\pi\sqrt{-\lambda}}) = D_2(1 - e^{-2\pi\sqrt{-\lambda}}).$$

We can combine the two equations in the system to deduce

$$\begin{aligned} D_1(1 - e^{2\pi\sqrt{-\lambda}}) &= -D_2(1 - e^{-2\pi\sqrt{-\lambda}}) \\ &= -D_1(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

Since we are currently in the case $\lambda < 0$, we have $1 - e^{2\pi\sqrt{-\lambda}} \neq 0$, and so we can divide $1 - e^{2\pi\sqrt{-\lambda}}$ from both sides of our previous equation to conclude $C_1 = -C_1$, or $C_1 = 0$. Likewise, we can combine the two equations in the system to deduce

$$\begin{aligned} D_2(1 - e^{-2\pi\sqrt{-\lambda}}) &= D_1(1 - e^{2\pi\sqrt{-\lambda}}) \\ &= -D_2(1 - e^{-2\pi\sqrt{-\lambda}}). \end{aligned}$$

Since we are currently in the case $\lambda < 0$, we have $1 - e^{-2\pi\sqrt{-\lambda}} \neq 0$, and so we can divide $1 - e^{-2\pi\sqrt{-\lambda}}$ from both sides of our previous equation to conclude $D_2 = -D_2$, or $D_2 = 0$. So we have

$$\begin{aligned} \Theta(\theta) &= C_1 e^{\sqrt{-\lambda}\theta} + C_2 e^{-\sqrt{-\lambda}\theta} \\ &= 0e^{\sqrt{-\lambda}\theta} + 0e^{-\sqrt{-\lambda}\theta} \\ &= 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} w(r,\theta) &= R(r)\Theta(\theta) \\ &= R(r) \cdot 0 \\ &= 0, \end{aligned}$$

which is a trivial solution.

- Case 2: Suppose $\lambda = 0$. Then we have

$$\Theta(\theta) = D_1\theta + D_2,$$
$$\Theta(0) = \Theta(2\pi),$$

which implies $D_1 = 0$, and so we have

$$\begin{aligned} \Theta(x) &= D_1\theta + D_2 \\ &= D_1 \cdot 0 + D_2 \\ &= D_2. \end{aligned}$$

The derivative is

$$\begin{aligned} \frac{d}{d\theta}\Theta(\theta) &= \frac{d}{d\theta}(D_2) \\ &= 0, \end{aligned}$$

which clearly satisfies $\frac{d}{d\theta}\Theta(0) = 0 = \frac{d}{d\theta}\Theta(2\pi)$. Next, observe that $\ln(r)$ is undefined at the origin (at $r = 0$). Following page 197 of the textbook, we only consider smooth solutions and disregard any solutions that are undefined at the origin, and so we shall impose the condition $C_1 = 0$. So we have

$$\begin{aligned} R(r) &= C_1 \ln(r) + C_2 \\ &= 0 \ln(r) + C_2 \\ &= C_2. \end{aligned}$$

Therefore, if we let $\frac{\alpha_0}{2} = C_2 D_2$, then we have

$$\begin{aligned} w_0(r,\theta) &= R(r)\Theta(\theta) \\ &= C_2 D_2 \\ &= \frac{\alpha_0}{2}, \end{aligned}$$

which is a nontrivial smooth solution.

- Case 3: Suppose $\lambda > 0$. Then we have

$$\begin{aligned} &\Theta(x) = D_1 \cos(\sqrt{\lambda}\theta) + D_2 \sin(\sqrt{\lambda}\theta), \\ &\Theta(0) = \Theta(2\pi), \end{aligned}$$

which implies

$$D_1 = D_1 \cos(2\pi\sqrt{\lambda}) + D_2 \sin(2\pi\sqrt{\lambda}). \tag{1}$$

We also have

$$\begin{aligned} \frac{d}{d\theta}\Theta(x) &= \sqrt{\lambda}(-D_1 \sin(\sqrt{\lambda}\theta) + D_2 \cos(\sqrt{\lambda}\theta)), \\ \frac{d}{d\theta}\Theta(0) &= \frac{d}{d\theta}\Theta(2\pi), \end{aligned}$$

which implies

$$D_2 = -D_1 \sin(2\pi\sqrt{\lambda}) + D_2 \cos(2\pi\sqrt{\lambda}). \tag{2}$$

Now, we claim that, if either $\sin(2\pi\sqrt{\lambda}) \neq 0$ or $\cos(2\pi\sqrt{\lambda}) \neq 1$, then we have $D_1 = 0$ and $D_2 = 0$.

  - Subcase 1: Suppose $\sin(2\pi\sqrt{\lambda}) \neq 0$. Multiply both sides of (1) by $-\cos(2\pi\sqrt{\lambda})$ and both sides of (2) by $\sin(2\pi\sqrt{\lambda})$ to obtain

$$\begin{aligned} -D_1 \cos(2\pi\sqrt{\lambda}) &= -D_1 \cos^2(2\pi\sqrt{\lambda}) - D_2 \sin(2\pi\sqrt{\lambda})\cos(2\pi\sqrt{\lambda}), \\ D_2 \sin(2\pi\sqrt{\lambda}) &= -D_1 \sin^2(2\pi\sqrt{\lambda}) + D_2 \cos(2\pi\sqrt{\lambda})\sin(2\pi\sqrt{\lambda}), \end{aligned}$$

from which we can add up both sides of the two equations to get

$$-D_1 \cos(2\pi\sqrt{\lambda}) + D_2 \sin(2\pi\sqrt{\lambda}) = -D_1. \tag{3}$$

We equate (1) and (3) to get

$$\cancel{D_1 \cos(2\pi\sqrt{\lambda})} - D_2 \sin(2\pi\sqrt{\lambda}) = \cancel{D_1 \cos(2\pi\sqrt{\lambda})} + D_2 \sin(2\pi\sqrt{\lambda}),$$

which simplifies to

$$-D_2 \cancel{\sin(2\pi\sqrt{\lambda})} = D_2 \cancel{\sin(2\pi\sqrt{\lambda})}.$$

Since we assumed $\sin(2\pi\sqrt{\lambda}) \neq 0$, we can divide both sides by $\sin(2\pi\sqrt{\lambda})$ to get $-D_2 = D_2$, which means $D_2 = 0$. Substitute $D_2 = 0$ into (2) to obtain

$$0 = -D_1 \sin(2\pi\sqrt{\lambda}),$$

which implies $D_1 = 0$ because, once again, we assumed $\sin(2\pi\sqrt{\lambda}) \neq 0$.

  - Subcase 2: Suppose $\cos(2\pi\sqrt{\lambda}) \neq 1$. Then we can rewrite (1) and (2) as

$$D_1(1 - \cos(2\pi\sqrt{\lambda})) = D_2 \sin(2\pi\sqrt{\lambda}), \tag{4}$$

$$D_2(1 - \cos(2\pi\sqrt{\lambda})) = -D_1 \sin(2\pi\sqrt{\lambda}),. \tag{5}$$

Multiply both sides of (4) by D_1 and both sides of (5) by D_2 to obtain

$$\begin{aligned} D_1^2(1 - \cos(2\pi\sqrt{\lambda})) &= D_1 D_2 \sin(2\pi\sqrt{\lambda}), \\ D_2^2(1 - \cos(2\pi\sqrt{\lambda})) &= -D_1 D_2 \sin(2\pi\sqrt{\lambda}), \end{aligned}$$

from which we can add up both sides of the two equations to get

$$(D_1^2 + D_2^2)(1 - \cos(2\pi\sqrt{\lambda})) = 0.$$

Since we assumed $\cos(2\pi\sqrt{\lambda}) \neq 1$, we must conclude $D_1^2 + D_2^2 = 0$, which forces $D_1 = 0$ and $D_2 = 0$.

So we have proved our claim. Now that we have established our claim, we would have

$$\begin{aligned}\Theta(\theta) &= D_1\cos(\sqrt{\lambda}x) + D_2\sin(\sqrt{\lambda}x)\\ &= 0\cos(\sqrt{\lambda}x) + 0\sin(\sqrt{\lambda}x)\\ &= 0,\end{aligned}$$

which would imply that $w(r, \theta) = R(r)\Theta(\theta)$ is a trivial solution. Therefore, to find a nontrivial solution for this case, we should assume both

$$\begin{aligned}\sin(2\pi\sqrt{\lambda}) &= 0,\\ 1 - \cos(2\pi\sqrt{\lambda}) &= 0,\end{aligned}$$

which imply $2\pi\sqrt{\lambda} = 2n\pi$, or equivalently

$$\lambda_n = \lambda = n^2,$$

and so we have

$$\begin{aligned}\Theta_n(\theta) &= D_{1,n}\cos(\sqrt{\lambda_n}\theta) + D_{2,n}\sin(\sqrt{\lambda_n}\theta)\\ &= D_{1,n}\cos(\sqrt{n^2}\theta) + D_{2,n}\sin(\sqrt{n^2}\theta)\\ &= D_{1,n}\cos(n\theta) + D_{2,n}\sin(n\theta)\end{aligned}$$

and

$$\begin{aligned}R_n(r) &= C_{1,n}r^{\sqrt{\lambda_n}} + C_{2,n}r^{-\sqrt{\lambda_n}}\\ &= C_{1,n}r^{\sqrt{n^2}} + C_{2,n}r^{-\sqrt{n^2}}\\ &= C_{1,n}r^n + C_{2,n}r^{-n}\end{aligned}$$

for $n = 1, 2, 3, \ldots$. Observe that r^{-n} for $n = 1, 2, \ldots$ is undefined at the origin (at $r = 0$). Following page 197 of the textbook, we only consider smooth solutions and disregard any solutions that are undefined at the origin, and so we shall impose the condition $C_{2,n} = 0$. So we have

$$\begin{aligned}R_n(r) &= C_{1,n}r^n + C_{2,n}r^{-n}\\ &= C_{1,n}r^n + 0r^{-n}\\ &= C_{1,n}r^n.\end{aligned}$$

Therefore, if we let $\alpha_n := C_{1,n}D_{1,n}$ and $\beta_n := C_{1,n}D_{2,n}$, then we have

$$\begin{aligned}w_n(r, \theta) &= R_n(r)\Theta_n(\theta)\\ &= (C_{1,n}r^n)(D_{1,n}\cos(n\theta) + D_{2,n}\sin(n\theta))\\ &= r^n(C_{1,n}D_{1,n}\cos(n\theta) + C_{1,n}D_{2,n}\sin(n\theta))\\ &= r^n(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta)).\end{aligned}$$

for $n = 1, 2, 3, \ldots$. This is a nontrivial smooth solution, as desired.

We recall that an addition of smooth solutions is again a smooth solution. So that means, as we have established already that each $w_n(r, \theta)$ is a nontrivial smooth solution for $n = 1, 2, 3\ldots$, it follows that

$$\begin{aligned}w(r, \theta) &= w_0(r, \theta) + \sum_{n=1}^{\infty} w_n(r, \theta)\\ &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta))\end{aligned}$$

is the general smooth solution of the Laplace equation. Next, we will compute the Fourier coefficients $\alpha_0, \alpha_n, \beta_n$. We have

$$w(1, \theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}(\alpha_n\cos(n\theta) + \beta_n\sin(n\theta))$$

and the given boundary condition

$$\begin{aligned}w(1, \theta) &= x^2 + y\\ &= \cos^2(\theta) + \sin(\theta)\\ &= \frac{1}{2}(1 + \cos(2\theta)) + \sin(\theta)\\ &= \frac{1}{2} + \frac{1}{2}\cos(2\theta) + \sin(\theta).\end{aligned}$$

Both our expressions of $u(x,0)$ yield

$$\frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)) = \frac{1}{2} + \frac{1}{2}\cos(2\theta) + \sin(\theta).$$

By the uniqueness of the Fourier series expansion, we can equate the terms of both sides of our above equation to find the Fourier coefficients

$$\begin{aligned} \alpha_0 &= 1, \\ \alpha_2 &= \frac{1}{2}, \\ \alpha_n &= 0 \end{aligned}$$

for $n = 1$ and for $n = 3, 4, 5, \ldots$ and

$$\begin{aligned} \beta_1 &= 1, \\ \beta_n &= 0 \end{aligned}$$

for $n = 2, 3, 4, \ldots$. Therefore, our formal solution in polar coordinates is

$$\begin{aligned} w(r,\theta) &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)) \\ &= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} \beta_n r^n \sin(n\theta) \\ &= \frac{\alpha_0}{2} + \left(\alpha_2 r^2 \cos(2\theta) + \sum_{\substack{n=1 \\ n=3,4,5,\ldots}} \alpha_n r^n \cos(n\theta) \right) + \left(\beta_1 r^1 \sin(1\theta) + \sum_{n=2,3,4,\ldots} \beta_n r^n \sin(n\theta) \right) \\ &= \frac{1}{2} + \left(\frac{1}{2} r^2 \cos(2\theta) + \sum_{\substack{n=1 \\ n=3,4,5,\ldots}} 0 r^n \cos(n\theta) \right) + \left(1 r^1 \sin(1\theta) + \sum_{n=2,3,4,\ldots} 0 r^n \sin(n\theta) \right) \\ &= \boxed{\frac{1}{2} + \frac{1}{2} r^2 \cos(2\theta) + r\sin(\theta)}. \end{aligned}$$

In Cartesian coordinates, our formal solution is

$$\begin{aligned} w(x,y) &= w(r,\theta) \\ &= \frac{1}{2} + \frac{1}{2} r^2 \cos(2\theta) + r\sin(\theta) \\ &= \frac{1}{2} + \frac{1}{2} r^2 (\cos^2(\theta) - \sin^2(\theta)) + r\sin(\theta) \\ &= \frac{1}{2} + \frac{1}{2} ((r\cos(\theta))^2 - (r\sin(\theta))^2) + r\sin(\theta) \\ &= \boxed{\frac{1}{2} + \frac{1}{2}(x^2 - y^2) + y}, \end{aligned}$$

where we used $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r^2 = x^2 + y^2$. □