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Spring 2020

Homework 15 solutions

*Note:* Almost all steps for solving an ordinary differential equation (for example, any material from MATH 046 at UC Riverside) are omitted from my solutions for purposes of brevity.

1. (a) Solve the problem

$$\begin{aligned} yu_x - xu_y + u_z &= u^2, \\ u(x, y, 0) &= \frac{1}{y}. \end{aligned}$$

*Hint:* Follow the worked examples in Lecture 16 (May 27). You may also refer to Example 2 of Lecture 4 (April 10).

*Solution.* Our characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= y, \\ \frac{\partial y}{\partial t} &= -x, \\ \frac{\partial z}{\partial t} &= 1, \\ \frac{\partial u}{\partial t} &= u^2 \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(0, s_1, s_2) &= s_1, \\ y(0, s_1, s_2) &= s_2, \\ z(0, s_1, s_2) &= 0, \\ u(0, s_1, s_2) &= u(x(0, s_1, s_2), y(0, s_1, s_2), z(0, s_1, s_2)) = u(s_1, s_2, 0) = \frac{1}{s_2}. \end{aligned}$$

First, we notice

$$\begin{aligned} \frac{\partial^2 x}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\right) \\ &= \frac{\partial y}{\partial t} \\ &= -x, \end{aligned}$$

which is equivalent to the second-order equation

$$\frac{\partial^2 x}{\partial t^2} + x = 0,$$

from which we can solve in $t$ to obtain

$$x(t, s_1, s_2) = C_1(s_1, s_2)\cos(t) + C_2(s_1, s_2)\sin(t),$$

where $C_1(s_1, s_2), C_2(s_1, s_2)$ are both constant in $t$. Applying the initial conditions

$$\begin{aligned} x(0, s_1, s_2) &= s_1, \\ \frac{\partial x}{\partial t}(0, s_1, s_2) &= y(0, s_1, s_2) = s_2 \end{aligned}$$

gives $C_1(s_1, s_2) = s_1$ and $C_2(s_1, s_2) = s_2$, and so we get

$$x(t, s_1, s_2) = s_1\cos(t) + s_2\sin(t).$$

Next, we notice

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) \\ &= \frac{\partial}{\partial t}(-x) \\ &= -\frac{\partial x}{\partial t} \\ &= -y, \end{aligned}$$

which is equivalent to the second-order equation

$$\frac{\partial^2 y}{\partial t^2} + y = 0,$$

from which we can solve in $t$ to obtain

$$y(t, s_1, s_2) = C_3(s_1, s_2)\cos(t) + C_4(s_1, s_2)\sin(t),$$

where $C_3(s_1, s_2), C_4(s_1, s_2)$ are both constant in $t$. Applying the initial conditions

$$\begin{aligned} y(0, s_1, s_2) &= s_2, \\ \frac{\partial y}{\partial t}(0, s_1, s_2) &= -x(0, s_1, s_2) = -s_1 \end{aligned}$$

gives $C_3(s_1, s_2) = s_2$ and $C_4(s_1, s_2) = -s_1$, and so we get

$$y(t, s_1, s_2) = s_2\cos(t) - s_1\sin(t).$$

The characteristic curves $x(t, s_1, s_2), y(t, s_1, s_2)$ are algebraically equivalent to

$$\begin{aligned} s_2\sin(t) &= x - s_1\cos(t), \\ s_2\cos(t) &= y + s_1\sin(t), \end{aligned}$$

respectively. Observe that we have

$$\begin{aligned} s_2 &= s_2(\cos^2(t) + \sin^2(t)) \\ &= (s_2\cos(t))\cos(t) + (s_2\sin(t))\sin(t) \\ &= (y - s_1\sin(t))\cos(t) + (x + s_1\cos(t))\sin(t) \\ &= y\cos(t) - \cancel{s_1\sin(t)\cos(t)} + x\sin(t) + \cancel{s_1\sin(t)\cos(t)} \\ &= y\cos(t) + x\sin(t). \end{aligned}$$

The third characteristic equation and the third initial condition

$$\begin{aligned} \frac{\partial z}{\partial t} &= 1, \\ z(0, s_1, s_2) &= 0 \end{aligned}$$

implies the characteristic curve

$$z(t, s_1, s_2) = t.$$

Finally, the fourth characteristic equation and the fourth initial condition

$$\begin{aligned} \frac{\partial u}{\partial t} &= u^2, \\ u(0, s_1, s_2) &= \frac{1}{s_2} \end{aligned}$$

implies the characteristic curve

$$u(t, s_1, s_2) = \frac{1}{s_2 - t}.$$

Our equations

$$\begin{aligned} y\cos(t) + x\sin(t) &= s_2, \\ z &= t \end{aligned}$$

can be solved simultaneously to obtain

$$\begin{aligned} t &= z, \\ s_2 &= y\cos(z) + x\sin(z). \end{aligned}$$

So our solution is

$$\begin{aligned} u(x, y, z) &= u(x(t, s_1, s_2), y(t, s_1, s_2), z(t, s_1, s_2)) \\ &= u(t, s_1, s_2) \\ &= \frac{1}{s_2 - t} \\ &= \boxed{\frac{1}{y\cos(z) + x\sin(z) - z}}, \end{aligned}$$

as desired. □

(b) Find the Jacobian on the initial surface at $t = 0$. Does the solution exist? If so, is it unique?

*Solution.* For this problem, the Jacobian is

$$\begin{aligned}
J|_{t=0} &= \begin{vmatrix} \frac{\partial}{\partial s_1}x(0,s_1,s_2) & \frac{\partial}{\partial s_1}y(0,s_1,s_2) & \frac{\partial}{\partial s_1}z(0,s_1,s_2) \\ \frac{\partial}{\partial s_2}x(0,s_1,s_2) & \frac{\partial}{\partial s_2}y(0,s_1,s_2) & \frac{\partial}{\partial s_2}z(0,s_1,s_2) \\ \frac{\partial}{\partial t}x(0,s_1,s_2) & \frac{\partial}{\partial t}y(0,s_1,s_2) & \frac{\partial}{\partial t}z(0,s_1,s_2) \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial s_1}x(t,s_1,s_2)|_{t=0} & \frac{\partial}{\partial s_1}y(t,s_1,s_2)|_{t=0} & \frac{\partial}{\partial s_1}z(t,s_1,s_2)|_{t=0} \\ \frac{\partial}{\partial s_2}x(t,s_1,s_2)|_{t=0} & \frac{\partial}{\partial s_2}y(t,s_1,s_2)|_{t=0} & \frac{\partial}{\partial s_2}z(t,s_1,s_2)|_{t=0} \\ \frac{\partial}{\partial t}x(t,s_1,s_2)|_{t=0} & \frac{\partial}{\partial t}y(t,s_1,s_2)|_{t=0} & \frac{\partial}{\partial t}z(t,s_1,s_2)|_{t=0} \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial s_1}(s_1\cos(t)+s_2\sin(t))|_{t=0} & \frac{\partial}{\partial s_1}(s_2\cos(t)-s_1\sin(t))|_{t=0} & \frac{\partial}{\partial s_1}(t)|_{t=0} \\ \frac{\partial}{\partial s_2}(s_1\cos(t)+s_2\sin(t))|_{t=0} & \frac{\partial}{\partial s_2}(s_2\cos(t)-s_1\sin(t))|_{t=0} & \frac{\partial}{\partial s_2}(t)|_{t=0} \\ \frac{\partial}{\partial t}(s_1\cos(t)+s_2\sin(t))|_{t=0} & \frac{\partial}{\partial t}(s_2\cos(t)-s_1\sin(t))|_{t=0} & \frac{\partial}{\partial t}(t)|_{t=0} \end{vmatrix} \\
&= \begin{vmatrix} \cos(t)|_{t=0} & -\sin(t)|_{t=0} & 0|_{t=0} \\ \sin(t)|_{t=0} & \cos(t)|_{t=0} & 0|_{t=0} \\ (-s_1\sin(t)+s_2\cos(t))|_{t=0} & (-s_2\sin(t)-s_1\cos(t))|_{t=0} & 1|_{t=0} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_2 & -s_1 & 1 \end{vmatrix} \\
&= 1 \\
&\neq 0,
\end{aligned}$$

meaning that the transversality condition holds for this problem. So we conclude from the existence-uniqueness theorem that the solution exists and is unique. □