MATH 146C discussion
University of California, Riverside

Ryan Ta
Spring 2020

Homework 3 solutions

Note: Any steps for solving an ordinary differential equation (for example, any material from MATH 046 at UC Riverside) are omitted from my solutions for purposes of brevity.

1. Solve the Cauchy problem

$$\begin{aligned} 2u_x + 3u_y &= u^2, \\ u(x,0) &= 1. \end{aligned}$$

Solution. We employ the method of characteristics for first-order partial differential equations. We parameterize the following variables:

$$\begin{aligned} &x = x(t,s), \\ &y = y(t,s), \\ &u = u(x,y) = u(x(t,s), y(t,s)) = u(t,s). \end{aligned}$$

Our characteristic equations are

$$\begin{aligned} \frac{\partial x}{\partial t} &= 2, \\ \frac{\partial y}{\partial t} &= 3, \\ \frac{\partial u}{\partial t} &= u^2 \end{aligned}$$

with the initial conditions

$$\begin{aligned} &x(0,s) = s, \\ &y(0,s) = 0, \\ &u(0,s) = u(x(0,s), y(0,s)) = u(s,0) = 1. \end{aligned}$$

We can solve the characteristic equations and apply the initial conditions to obtain the characteristic curves

$$\begin{aligned} x(t,s) &= 2t + s, \\ y(t,s) &= 3t, \\ u(t,s) &= \frac{1}{1-t}. \end{aligned}$$

The first two characteristic curves $x(t,s), y(t,s)$ imply

$$\begin{aligned} s &= x - 2t, \\ t &= \frac{y}{3}. \end{aligned}$$

So our solution is

$$\begin{aligned} u(x,y) &= u(t,s) \\ &= \frac{1}{1-t} \\ &= \frac{1}{1-\frac{y}{3}} \\ &= \boxed{\frac{3}{3-y}}, \end{aligned}$$

as desired. □

Remark. *Notice here that we did not even use s to find our expression of $u(x,y)$ at all! Also notice that the solution $u(x,y) = \frac{3}{3-y}$ depends only on y; it is constant with respect to x. We only needed to eliminate the parameter t for this problem. But in general, you need to eliminate both the parameters t and s to obtain your solution $u(x,y)$, which in general depends on both x and y.*