MATH 146C discussion Ryan Ta
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Homework 5 solutions

Note: Almost all steps for solving an ordinary differential equation (for example, any material from MATH 046 at UC Riverside) are omitted from my solutions for purposes of brevity.

1. Consider the problem

$$u_x + 2u_y = u^2,$$
$$u(x, 2x) = 1.$$

(a) Find the characteristic curves. What are the projections of the characteristic curves on the (x, y) plane?

Solution. Our characteristic equations are

$$\frac{\partial x}{\partial t} = 1,$$
$$\frac{\partial y}{\partial t} = 2,$$
$$\frac{\partial u}{\partial t} = u^2$$

with the initial conditions

$$x(0, s) = s,$$
$$y(0, s) = 2x(0, s) = 2s,$$
$$u(0, s) = u(x(0, s), y(0, s)) = u(s, 2s) = 1.$$

We can solve the characteristic equations and apply the initial conditions to obtain the characteristic curves

$$x(t, s) = t + s,$$
$$y(t, s) = 2t + 2s,$$
$$u(t, s) = \frac{1}{1 - t}.$$

Now, we have

$$\frac{dy}{dx} = \frac{\frac{\partial y}{\partial t}}{\frac{\partial x}{\partial t}} = \frac{2}{1} = 2,$$

which implies

$$\boxed{y = 2x + C},$$

where C is a constant. □

(b) Discuss the existence and uniqueness of the solution.

Proof. We claim that there does not exist a solution to the problem. Suppose instead that u is a solution to the problem. Then, from our solution to part (a), we have

$$u(t, s) = \frac{1}{1 - t},$$

which implies

$$\frac{\partial u}{\partial s} = \frac{\partial}{\partial s}\left(\frac{1}{1 - t}\right) = 0.$$

On the other hand, by the chain rule, we also have

$$\frac{\partial u}{\partial s} = u_x \frac{\partial x}{\partial s} + u_y \frac{\partial y}{\partial s} = u_x \frac{\partial}{\partial s}(s) + u_y \frac{\partial}{\partial s}(2s) = u_x + 2u_y = u^2.$$

We can equate our two expressions of $\frac{\partial u}{\partial s}$ to conclude $u^2 = 0$, which is a contradiction if $u(x, y) \neq 0$. So there does not exist a nontrivial solution of the problem. And, if $u(x, y) = 0$, then in particular we would have $u(x, 2x) = 0$, which contradicts the given condition $u(x, 2x) = 1$. So there also does not exist a trivial solution of the problem. We conclude that there exists no solution of this problem, as we claimed. $\square$